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Boundary Value Problem of Nonlinear Fractional Differential Equations of Mixed Volterra-Fredohlm Integral Equations in Banach Space

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Abstract: This paper is dedicated to study the existence and uniqueness of solutions for a boundary value problem of nonlinear fractional differential equations of mixed Volterra-Fredohlm integral equations in Banach space, the recent researches considered the study of differential equations of mixed Volterra-Fredholm integral equations with classical order and the study of existence and uniqueness of solutions using approched numerical methodes, the objective of this paper is the study of the existence and uniqueness of fractional order of differential equations with mixed Volterra-Fredholm integral equations using fixed point theory. This work have two important results, the first result was the discussion of the existence of solutions using the Krasnoselskii fixed point theorem after transforming the problem into integral equation firstly then into operator problem suitable for the fixed point theory. The second result will be the existence and uniqueness of solution, this result was obtained by the use of Banach fixed point theorem after the same transformation used in the first one. This work give as conclusion that the boundary value problem of nonlinear fractional differential equations of mixed Volterra-Fredholm integral equation has a unique solution in Banach space. Finally, this work was ended with an example to illustrate the results obtained.

Keywords: Mixed Volterra-Fredohlm Integral Equation, Existence and Uniqueness, Fixed Point Theory

1. Introduction

In recent years, fractional calculus has attracted a large number of mathematicians and modelers. many researchers focused on the divaloping of the topic of fractional order initial and boundary value problems. This is result of the position of mathematical models associated with physical problems based on fractional differential equations subject to initial and boundary value preblems. Some fractional order initial value problems and boundary value problems, involving Riemann-Liouville, Liouville, Caputo and Hadamard type fractional differential equations, has attracted the attention of many researchers, for instance, see [1-5].

The importance and the value of the subject of mixed Volterra-Fredholm integral equations makes this subject a subject of considerable interest. Studies on population dynamics, parabolic boundary value problems, the mathematical modeling of the spatio-temporal development of an epidemic and various physical and biological models lead to the mixed Volterra-Fredholm integral equations.

Mirzaee and Hoseini [6] discussed the solution of The linear mixed Volterra-Fredholm integral equation using Fibonacci collocation method, while the numerical solution via modification of the hat function was presented in [7]. Hasan and Sulaiman [8] studied the existence and uniqueness results for the linear mixed Volterra-Fredholm integral equation is given by

$$u(x) = f(x) + \lambda \int_a^x \int_a^b k(r,t)u(t)dtdr$$

Several other fixed point type and numerical methods were discussed in the literature, see for more [9-12] and the references therein.

The academic value and importance of the topic of fractional differential equations and systems in the modeling of so many phenomina, and the studies and researches published in this field gives motivation to study such model. As we know, there are few research on the topic of fractional differential equations with mixed Volterra-Fredholm integral equations. Motivated by the researches cited above, this paper will consider the study a new type of fractional differential equations, and a new type of hybrid perturbation (we perturbate our boundary value problem with Volterra-Fredholm integrql equation)

In this paper, we study the existence and uniqueness of solution for a boundary value problem of nonlinear fractional differential equations of mixed Volterra-Fredohlm integral equations in Banach space, consider the follwing problem

$$\begin{aligned} {}^cD^{\alpha}u(x) = \quad & f(x,u(x)) + \lambda \textstyle\int_0^x \int_0^1 k(r,t)u(t)dtdr \qquad (1) \\ & u(0) = u(1) = 0, \end{aligned}$$

where $f(x,u(x))$ is a known continuous function on the interval $[0,1]$, the kernel $k(r,t)$ is known and continuous on the region $D = \{(r,t); 0 \le t \le 1 \& 0 \le r \le x \le 1\}$, while $u(x)$ is unknown continuous function to be determined.

This work is organized as follows. Section 2 contains the preliminary (Definitions, Lemmas, Theorems) that we need in the proof of results of this work. Section 3, focus on the presentation of the solution for the boundary value problem for nonlinear fractional differential equations of mixed Volterra-Fredohlm integral equations (1) involving the Caputo fractional derivative, then the proof of the existence results will be given in several steps. Section 4. Contain the proof of the existence and uniqueness of solution to the boundary value problem in two steps. Finally, an example will be added to illustrate and clear the results obtained.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions from fractional calculus theory. These definitions and properties can be found in the recent monograph [13-17]

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \mapsto \mathbb{R}$ is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, \qquad (2)$$

provided that the right-hand side is defined pointwise, where $\Gamma(.)$ is the Gamma function.

Given a continuous function $f : (0,\infty) \longrightarrow \mathbb{R}$, its fractional derivative with order $\alpha > 0$ in the sense of Riemann-Liouville, is given

Definition 2.2.

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)\, ds, \qquad (3)$$

where $n = [\alpha] + 1$.

Lemma 2.1. [1] Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$. Then

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2..., n$, where $n = [\alpha] + 1$

Lemma 2.2 (Ascoli-Arzela theorem). A be a subset of $C(J,E)$, A is relatively compact in $C(J,E)$ if and only if the following conditions are checked:

(i) The unit A is limited.
$\exists k > 0$ such that $\|f(x)\|_E \le k$ for $x \in J$ and $f \in A$.

(ii) Unit A is equicontinuous.
$\forall \varepsilon > 0, \exists \delta > 0$ and for evry $t_1, t_2 \in J$ we have $|t_1, t_2| < \delta \Rightarrow \|f(t_1) - f(t_2)\|_E < \varepsilon$.

(iii) For any $x \in J$ the unit $\{f(x), f \in A\} \subset E$ is relatively compact.

Lemma 2.3 (Banach fixed point theorem). Let X be a non-empty complete metric space, and $T : X \mapsto X$ is a contraction mapping. Then, there exists a unique point $x \in X$ such that $Tx = x$.

Lemma 2.4 (Krasnoselskii fixed point theorem). Let E be a non-empty, bounded, closed and convex subset in Banach space X: If $A, B : E \mapsto E$satisfy the following assumptions:

1. $Ax + By \in E$, for every $x, y \in X$,
2. A is a contraction,
3. B is compact and continuous.

Then, there exists $z \in X$ such that $Az + Bz = z$.

3. Existence Results

This section consider the study of the existence of solutions to the boundary value problem (1). By Lemma 3.1, the boundary value problem (1) be transformed into a fixed point problem.

Let $F \in C(0,1)$ and $u \in C(J.\mathbb{R})$ be continuous real valued functions, then the solution of the problem

Lemma 3.1.

$$\begin{aligned} {}^cD^{\alpha}u(x) = \quad & F(x) + \lambda \textstyle\int_0^x \int_0^1 k(r,t)u(t)dtdr, \qquad (4) \\ & u(0) = u(1) = 0, \end{aligned}$$

is given by

$$u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}F(s)ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds.$$
$$-\left[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}F(s)ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right]x. \quad (5)$$

Proof Apply the Riemann-Liouville fractional integral I^α on the both side of equation (2), and using Lemma 2.1, we get

$$u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}F(s)ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds + c_1x + c_2. \quad (6)$$

The boundary condition $u(0) = 0$ gives $c_2 = 0$.
using the boundary value condition $u(1) = 0$,

$$c_1 = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}F(s)ds - \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds,$$

taking the values c_1 and c_2 into (4) gets

$$u(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}F(s)ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds.$$
$$-\left[\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}F(s)ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right]x.$$

Thus complete the proof.

The first result concerns the study of existence of solution for the boundary value problem (1) by using the Krasnoselskii fixed-point theorem. For this fact, let supose the followng assumptions about the functions previously defined.

Denote $X = (C([0,1]\times\mathbb{R})), \mathbb{R})$, the Banach space endowed with the norm

$$\|u\| = \sup_{t\in[0,1]} |u(t)|,$$

for $u \in X$.

$(\mathbf{H_1})$ Since the kernel is continuous on bounded interval, then

$$|k(r,t)| \leq M.$$

$(\mathbf{H_2})$ The function $f : J \times \mathbb{R} \mapsto \mathbb{R}$ is continuous and there exist non negative function σ such that

$$|f(x,u(x))| \leq \sigma(x).$$

$(\mathbf{H_3})$ There exist a positive constant C such that

$$|f(x,u) - f(x,v)| \leq C\|u-v\|$$

Theorem 3.1. Assume that the assumptions $H(1) - H(3)$ hold. If

$$\left(\frac{C}{\Gamma(\alpha+1)} + \frac{|\lambda| M}{\Gamma(\alpha+2)}\right) < 1.$$

then the mixed Volterra-Fredholm fractional differential equation (1) has at least one solution in X on J.

Proof

We define the operators $A, B : X \mapsto X$ by

$$Au(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}f(s,u(s))ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds,$$

and

$$Bu(x) = -\frac{x}{\Gamma(\alpha)}\left[\int_0^1 (1-s)^{\alpha-1}f(s,u(s))ds + \lambda\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right].$$

Now, we show that the operators A and B satisfy all the conditions of Lemma 2,4 in a series of steps.

Step 1. We define the set

$$S = \{u \in X, \|u\|_X \leqslant r,\}$$

where r is a positive real constant such that

$$r > \frac{2\|\sigma\|(\alpha+1)}{\Gamma(\alpha+2) - 2|\lambda| M}$$

First, we show that $Au + Bu \in S$. So for $u \in S$ and $x \in J$, we have

$$\begin{aligned}
& |Au(x) + Bu(x)| \\
= & \Big|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s,u(s))ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds \\
& -\frac{x}{\Gamma(\alpha)}\Big[\int_0^1 (1-s)^{\alpha-1} f(s,u(s))ds + \lambda\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\Big]\Big| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|f(s,u(s))|ds + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)||u(t)|dtdrds \\
& +\frac{x}{\Gamma(\alpha)}\Big[\int_0^1 (1-s)^{\alpha-1}|f(s,u(s))|ds + |\lambda|\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)||u(t)|dtdrds\Big] \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\sigma(s)ds + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 M|u(t)|dtdrds \\
& +\frac{x}{\Gamma(\alpha)}\Big[\int_0^1 (1-s)^{\alpha-1}\sigma(s)ds + |\lambda|\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 M|u(t)|dtdrds\Big] \\
\leq & \frac{\|\sigma\|}{\Gamma(\alpha+1)} + \frac{|\lambda| M\|u\|_X}{\Gamma(\alpha+2)} + \frac{\|\sigma\|}{\Gamma(\alpha+1)} + \frac{|\lambda| M\|u\|_X}{\Gamma(\alpha+2)} \\
\leq & 2\left(\frac{\|\sigma\|}{\Gamma(\alpha+1)} + \frac{|\lambda| M\|u\|_X}{\Gamma(\alpha+2)}\right) \\
\leq & \frac{2\|\sigma\|(\alpha+1)}{\Gamma(\alpha+2) - 2|\lambda| M} \leq r.
\end{aligned}$$

Therefore, $\|Au + Bu\|_X \leq r$, that means that $Au(x) + Bu(x) \in S$.
Step 2. Show that the operator B is contraction on S, for $u, v \in S$ and $x \in J$. Using the assumption $(H3)$ we have

$$\begin{aligned}
& |Bu(x) + Bv(x)| \\
= & \Big|\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,u(s))ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds \\
& -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,v(s))ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)v(t)dtdrds\Big| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}|f(s,u(s)) - f(s,v(s))|ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)||u(t)-v(t)|dtdrds \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}C|u(s)-v(s)|ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\int_0^s\int_0^1 M|u(t)-v(t)|dtdrds \\
\leq & \frac{C}{\Gamma(\alpha+1)}\|u(x)-v(x)\| + \frac{|\lambda| M}{\Gamma(\alpha+2)}\|u(x)-v(x)\| \\
\leq & \left(\frac{C}{\Gamma(\alpha+1)} + \frac{|\lambda| M}{\Gamma(\alpha+2)}\right)\|u(x)-v(x)\|.
\end{aligned}$$

Then, the operator A is a contraction on S.
Step 3. Now, prove that A is completely continuous on S. This needs to show that the set (AS) is uniformly bounded, the set $\overline{(AS)}$ is equicontinuous, and the operator $A : S \longmapsto S$ is continuous.
For $u \in S$ and $x \in J$,

$$\begin{aligned}
|Au(x)| = & \Big|\frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1} f(s,u(s))ds + \frac{\lambda}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\Big| \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}|f(s,u(s))|ds + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)||u(t)|dtdrds \\
\leq & \frac{1}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\sigma(s)ds + \frac{|\lambda|}{\Gamma(\alpha)}\int_0^x (x-s)^{\alpha-1}\int_0^s\int_0^1 M|u(t)|dtdrds \\
\leq & \frac{\|\sigma\|}{\Gamma(\alpha+1)} + \frac{|\lambda| M\|u\|_X}{\Gamma(\alpha+2)} \\
\leq & \frac{\|\sigma\|}{\Gamma(\alpha+1)} + \frac{|\lambda| Mr}{\Gamma(\alpha+2)}.
\end{aligned}$$

Then, the set (AS) is uniformly bounded.

Now, show that $\overline{(AS)}$ is equicontinuous. Let $x_1, x_2 \in J$ with $x_1 < x_2$, we have for any $u \in S$,

$$\begin{aligned}
& |Au(x_2) - Au(x_1)| \\
= & \left| \frac{1}{\Gamma(\alpha)} \int_0^{x_2} (x_2 - s)^{\alpha-1} f(s, u(s)) ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^{x_2} (x_2 - s)^{\alpha-1} \int_0^s \int_0^1 k(r,t) u(t) dt dr ds \right. \\
& \left. - \frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_1 - s)^{\alpha-1} f(s, u(s)) ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^{x_1} (x_1 - s)^{\alpha-1} \int_0^s \int_0^1 k(r,t) u(t) dt dr ds \right| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^{x_2} (x_2 - s)^{\alpha-1} |f(s, u(s))| ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{x_2} (x_2 - s)^{\alpha-1} \int_0^s \int_0^1 |k(r,t)| |u(t)| dt dr ds \\
& - \frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_1 - s)^{\alpha-1} |f(s, u(s))| ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{x_1} (x_1 - s)^{\alpha-1} \int_0^s \int_0^1 |k(r,t)| |u(t)| dt dr ds \\
\leq & \frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_2 - s)^{\alpha-1} |f(s, u(s))| ds + \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} |f(s, u(s))| ds \\
& + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{x_1} (x_2 - s)^{\alpha-1} \int_0^s \int_0^1 |k(r,t)| |u(t)| dt dr ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} \int_0^s \int_0^1 |k(r,t)| |u(t)| dt dr ds \\
& - \frac{1}{\Gamma(\alpha)} \int_0^{x_1} (x_1 - s)^{\alpha-1} |f(s, u(s))| ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_0^{x_1} (x_1 - s)^{\alpha-1} \int_0^s \int_0^1 |k(r,t)| |u(t)| dt dr ds \\
\leq & \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} |f(s, u(s))| ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} \int_0^s \int_0^1 |k(r,t)| |u(t)| dt dr ds \\
\leq & \frac{1}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} \sigma(s) ds + \frac{|\lambda|}{\Gamma(\alpha)} \int_{x_1}^{x_2} (x_2 - s)^{\alpha-1} \int_0^s \int_0^1 M |u(t)| dt dr ds \\
\leq & \frac{\|\sigma\|}{\Gamma(\alpha+1)} (x_2 - x_1)^{\alpha} - \frac{|\lambda| Mr}{\Gamma(\alpha+1)} (x_2 - x_1)^{\alpha} + \frac{|\lambda| Mr}{\Gamma(\alpha+2)} (x_2 - x_1)^{\alpha+1} \\
\leq & \frac{(x_2 - x_1)^{\alpha}}{\Gamma(\alpha+1)} (\|\sigma\| - |\lambda| Mr) + \frac{|\lambda| Mr}{\Gamma(\alpha+2)} (x_2 - x_1)^{\alpha+1}.
\end{aligned}$$

As $x_1 \mapsto x_2$ the right hand side of the above inequality tend to zero. Therefore, it follows that $\overline{(AS)}$ is equicontinuous.

Finally, show that the operator A is continuous in X. Let (u_n) be a sequence in S converging to a point $u \in S$. Then, by Lebesgue dominated convergence theorem, for all $x \in J$, we have

$$\begin{aligned}
& \lim_{n \mapsto \infty} Au_n(x) \\
= & \lim_{n \mapsto \infty} \left[\frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} f(s, u_n(s)) ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \int_0^s \int_0^1 k(r,t) u_n(t) dt dr ds \right] \\
= & \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \lim_{n \mapsto \infty} f(s, u_n(s)) ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \int_0^s \int_0^1 k(r,t) \lim_{n \mapsto \infty} u_n(t) dt dr ds \\
= & \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} f(s, u(s)) ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \int_0^s \int_0^1 k(r,t) u(t) dt dr ds = Au(x).
\end{aligned}$$

Consequently, A is continuous.

Therefore, A is also relatively compact on S. Using the Arzila-Ascolii's theorem, we conclude that A is compact on S. Now, all conditions of Krasnoselskii's fixed point theorem are satisfied, so the operator equation $Au+Bu$ has a fixed point on S. Finally, we deduce that the boundary value problem (1) has at least one solution in X on J.

4. Existence and Uniqueness Results

This section is for the study of the existence and uniqueness of solution of the the boundary value problem (1). This result is obtained by using the Banach fixed point theorem.

Theorem 4.1. Assume that the hypothesis (H_1), (H_2) and (H_3) are true. If

$$2 \left(\frac{C}{\Gamma(\alpha+1)} + \frac{|\lambda| M}{\Gamma(\alpha+2)} \right) < 1.$$

Then the boundary value problem (1) of fractional differential equation of Volterra-Fredohlm integral equations has a unique solution in X on J.

Proof Define the operator $T : X \mapsto X$ associated with our boundary value problem (1) by

$$Tu(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} f(s, u(s)) ds + \frac{\lambda}{\Gamma(\alpha)} \int_0^x (x - s)^{\alpha-1} \int_0^s \int_0^1 k(r,t) u(t) dt dr ds$$

$$-\frac{x}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}f(s,u(s))ds+\lambda\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right].$$

Now, the proof that the operator T has a fixed point in S which represents the unique solution of the boundary value problem (1) will be given in two steps..

Step 1. In the same set S defined in the first part.

First, let show that T maps S into itself ($TS \subset S$). For $x \in J$ and $u \in S$,

$$\begin{aligned}
&|Tu(x)|\\
=\ & \left|\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}f(s,u(s))ds+\frac{\lambda}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right.\\
&\left.-\frac{x}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}f(s,u(s))ds+\lambda\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right]\right|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left|f(s,u(s))\right|ds+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)|\,|u(t)|\,dtdrds\\
&+\frac{x}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}\left|f(s,u(s))\right|ds+|\lambda|\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)|\,|u(t)|\,dtdrds\right]\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\sigma(s)ds+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\int_0^s\int_0^1 M\,|u(t)|\,dtdrds\\
&+\frac{1}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}\sigma(s)ds+|\lambda|\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 M\,|u(t)|\,dtdrds\right]\\
\leq\ & \frac{\|\sigma\|}{\Gamma(\alpha+1)}+\frac{|\lambda|\,Mr}{\Gamma(\alpha+2)}+\frac{\|\sigma\|}{\Gamma(\alpha+1)}+\frac{|\lambda|\,Mr}{\Gamma(\alpha+2)}\\
\leq\ & 2\left(\frac{\|\sigma\|}{\Gamma(\alpha+1)}+\frac{|\lambda|\,Mr}{\Gamma(\alpha+2)}\right)\\
\leq\ & \frac{2\,\|\sigma\|\,(\alpha+1)}{\Gamma(\alpha+2)-2\,|\lambda|\,M}\leq r,
\end{aligned}$$

for any $u \in S$, we have

$$\|Tu(x)\| \quad\leq\quad \frac{2\,\|\sigma\|\,(\alpha+1)}{\Gamma(\alpha+2)-2\,|\lambda|\,M}\leq r,$$

which show that the operator T maps S into itself.

Step 2. Now, let show that the operator $T : S \mapsto S$ is a contraction. Let $u, v \in X$, and $x \in J$. By assumption (H_3), obtain

$$\begin{aligned}
&\|Tu(x)-Tv(x)\|\\
=\ & \left|\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}f(s,u(s))ds+\frac{\lambda}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right.\\
&-\frac{x}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}f(s,u(s))ds+\lambda\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)u(t)dtdrds\right]\\
&-\frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}f(s,v(s))ds-\frac{\lambda}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)v(t)dtdrds\\
&\left.+\frac{x}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}f(s,v(s))ds+\lambda\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 k(r,t)v(t)dtdrds\right]\right|\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\left|f(s,u(s))-f(s,v(s))\right|ds+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)|\,|u(t)-v(t)|\,dtdrds\\
&+\frac{x}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}\left|f(s,u(s))-f(s,v(s))\right|ds+|\lambda|\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 |k(r,t)|\,|u(t)-v(t)|\,dtdrds\right]\\
\leq\ & \frac{1}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}C\,|u(s)-v(s)|\,ds+\frac{|\lambda|}{\Gamma(\alpha)}\int_0^x(x-s)^{\alpha-1}\int_0^s\int_0^1 M\,|u(t)-v(t)|\,dtdrds\\
&+\frac{1}{\Gamma(\alpha)}\left[\int_0^1(1-s)^{\alpha-1}C\,|u(s)-v(s)|\,ds+|\lambda|\int_0^1(1-s)^{\alpha-1}\int_0^s\int_0^1 M\,|u(t)-v(t)|\,dtdrds\right]\\
\leq\ & \frac{C}{\Gamma(\alpha+1)}\|u-v\|+\frac{|\lambda|\,M}{\Gamma(\alpha+2)}\|u-v\|+\frac{C}{\Gamma(\alpha+1)}\|u-v\|+\frac{|\lambda|\,M}{\Gamma(\alpha+2)}\|u-v\|\\
\leq\ & 2\left(\frac{C}{\Gamma(\alpha+1)}+\frac{|\lambda|\,M}{\Gamma(\alpha+2)}\right)\|u-v\|\,.
\end{aligned}$$

This implies that the operator T is contraction.

Then by the Banach fixed point theorem, there exists a unique point $u \in X$ such that $Tu = u$, it is the unique solution of our boundary value problem (1). The proof of theorem 4.1 is completed.

5. Example

Consider the following boundary value problem of nonlinear fractional differential equation of mixed Volterra-Fredohlm integral equation

$$\begin{aligned} {}^cD^\alpha u(x) = \quad & f(x,u(x)) + \tfrac{1}{4}\textstyle\int_0^x \int_0^1 (r-t)u(t)dtdr \\ & u(0) = u(1) = 0, \end{aligned} \tag{7}$$

The boundary value problem (7) is a particular case of our problem (1) with $\alpha = \frac{1}{12}$, $\lambda = \frac{1}{4}$, $f(x,u(x)) = \frac{xu(x)\cos(u(x))}{12}$ and $k(r,t) = r - t$.

Clearly, f and k are contunious functions satisfied the assumption (H_1) and (H_2) with

$$\sigma(x) = \frac{x}{12},$$

and

$$|h(r,t)| \leq 1.$$

Also

$$\begin{aligned} |f(x,u) - f(x,v)| &\leq \left| \frac{xu(x)\cos(u(x))}{12} - \frac{xv(x)\cos(v(x))}{12} \right| \\ &\leq \frac{x}{12}\|u - v\| \\ &\leq \frac{1}{12}\|u - v\|. \end{aligned}$$

Using the values we have, we get

$$\left(\frac{C}{\Gamma(\alpha+1)} + \frac{|\lambda| M}{\Gamma(\alpha+2)} \right) = 0.1969230769 < 1.$$

Where $C = \frac{1}{12}$, $\lambda = \frac{1}{4}$ and $M = 1$.

Since the assumptions $(H_1), (H_2)$ and (H_3) hold, according to Theorem 3.1 the problem (7) has at least one solution.

To see if the solution is unique, note that assumptions $(H_1), (H_2)$ and (H_3) are hold, from first part of existence results. Also, the condition of Theorem 3.2

$$2\left(\frac{C}{\Gamma(\alpha+1)} + \frac{|\lambda| M}{\Gamma(\alpha+2)} \right) = 0.3938461538 < 1,$$

are satisfied, therefore from Theorem 3.2 the problem (7) has a unique solution.

6. Results

This paper cotained two important results, the first one based on Krasnoselskii fixed point theorem, after transforming the boundary value problem into integral equation and defined operator equation, then by applying Krasnoselskii fixed point theorem to get the existence resutl. The second result was the existence and uniqueness of solution for the boundary value problem, this result was proved by using the Banach fixed point theorem.

This work gives as results the existence and uniqueness of solution for boundary value problem of nonlinear fractional differential equations of mixed Volterra-Fredohlm integral equations in Banach space, using the fixed point theory.

7. Discussion

In view of the research cited in the introduction we can see that the problem we studied in this paper is new subject for Volterra-Fredohlm integral equations, this studies based on fractional order, more than this we studied our problem using fixed point theory, previous research studied this kind of problem uding numerical methodes. The different also in the hybrid perturbation (the boundary value problem was perturbated with Volterra-Fredholm integrql equation).

8. Conclusions

This work consider the existence and uniqueness of solutions for the boundary value problem for nonlinear fractional differential equations of mixed Volterra-Fredohlm integral equations in Banach space. The existence results of solutions for the boundary value problem (1) was obtained by transforming the problem into a Volttera integral equation and using the Krasnoselskii fixed point theorem under some conditions. The second result was the existence and uniqueness of solution for the boundary value problem, by using the Banach fixed point theorem and after transforming the problem into a fixed point problem, the result was proved.

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Conflicts of Interest

The authors declare no conflicts of interest.

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