

Algebraic Points of Degree at Most 3 on the Affine Equation Curve $y^{11} = x^4(x - 1)^4$

Mouhamadou Diaby Gassama*, Oumar Sall

Mathematics and Applications Laboratory, Faculty of Science and Technology, Assane Seck University in Ziguinchor, Ziguinchor, Senegal

Email address:

m.gassama1299@zig.univ.sn (Mouhamadou Diaby Gassama), osall@univ-zig.sn (Oumar Sall)

*Corresponding author

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Abstract: The quotients of Fermat curves $C_{r,s}(p)$ are studied by Oumar SALL. Among these studies are the cases $C_{r,s}(11)$ for $r = s = 1$. Mamina COLY and Oumar SALL have explicitly determined the algebraic points of degree at most 3 on \mathbb{Q} for the cases $C_{r,s}(11)$ for $r = s = 2$. Our work focuses on determining explicitly the algebraic points of degree at most 3 on \mathbb{Q} on the curve $C_{4,4}(11)$ which is a special case of Fermat quotient curves. Our study concerns the cases $C_{r,s}(11)$ for $r = s = 4$. It seems that the finiteness of the Mordell-Weil group of rational points of the Jacobien $J_{4,4}(11)(\mathbb{Q})$ is an essential condition. So to determine the algebraic points on the curve $C_{4,4}(11)$ we need a finiteness of the Mordell-Weill group of rational points of the Jacobien $J_{4,4}(11)(\mathbb{Q})$. The Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of rational points of the Jacobien is finite according to Faddev. Our note is in this framework. Our essential tools in this note are the Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of the Jacobien of $C_{4,4}(11)$ the Abel-Jacobi theorem and the study of linear systems on the curve $C_{4,4}(11)$. The result obtained concerns some quotients of Fermat curves. Indeed, the curve $C_{4,4}(11)$ which is the subject of our study, the set of algebraic points of degree at most 3 on \mathbb{Q} has been determined in an explicit way, to achieve this we have determined the quadratic points on the curve $C_{4,4}(11)$ on \mathbb{Q} and the cubic points on the curve $C_{4,4}(11)$ on \mathbb{Q} .

Keywords: Planes Curves, Degree of Algebraic Points, Jacobien

1. Introduction

Let C be a smooth algebraic curve defined on \mathbb{Q} . Let K be a field of numbers we note $\mathcal{C}(K)$ the set of points on C with coordinates in K , and $\bigcup_{[K:\mathbb{Q}] \leq d} \mathcal{C}(K)$ the set of points on C with coordinates in K of degree at most d on \mathbb{Q} . The degree of a point R of C algebraic on \mathbb{Q} is defined as the degree of its defining field on \mathbb{Q} ; in other words $deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$.

In this note we will focus on the curve $C_{4,4}(11)$ with affine equation $y^{11} = x^4(x - 1)^4$ which is a special case of quotients of Fermat curves $C_{r,s}(p) : y^p = x^r(x - 1)^s$, $1 \leq r, s$; $r + s \leq p - 1$ studied in [8-10]. The cases $C_{r,s}(11)$ for $r = s = 2$ are studied

in [2]. See [1, 3, 4, 11] for other explicit examples. Indeed, C corresponds to the curve $C_{4,4}(11)$. The curves $C_{r,s}(p)$ are quotients of F_p [6, 15].

We denote by $J_{4,4}(11)$ the Jacobien of $C_{4,4}(11)$ and by $j(P)$ the class denoted $[P - P_\infty]$ of $P - P_\infty$, that is to say j is the Jacobien fold $C_{4,4}(11) \rightarrow J_{4,4}(11)$. The Mordell-Weil group $J_{4,4}(11)(\mathbb{Q})$ of the rational points of the Jacobien is finite [5, 6, 12, 13]. The curve $C_{4,4}(11)$ in projective is $C_{4,4}(11) : Y^{11} = X^4 Z^7 (X - Z)^4$. Let us note P_0, P_1 and P_∞ the points defined by: $P_0 = (0, 0, 1)$; $P_1 = (1, 0, 1)$ and $P_\infty = (1, 0, 0)$. It follows from the work of Gross-Rohrlich in [6], that

$$\bigcup_{[K:\mathbb{Q}] \leq 2} C_{1,1}(11)(K) = \left\{ \left(\frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}}, y \right) \mid y \in \mathbb{Q} \right\} \cup \{ P_\infty \}$$

In this note we determine the set

$$\bigcup_{[K:\mathbb{Q}] \leq 3} \mathcal{C}_{4,4}(11)(K)$$

Our main result is the following: *Theorem 1.1* The set of algebraic points of degree at most 3 on \mathbb{Q} of the curve $\mathcal{C}_{4,4}(11)$ is given by:

$$\bigcup_{[K:\mathbb{Q}] \leq 3} \mathcal{C}_{4,4}(11)(K) = \{P_0, P_1, P_\infty\} \cup S_0 \cup S_1$$

with

$$S_0 = \left\{ \left(x, (\alpha x(x-1))^{\frac{1}{3}} \right) \mid \alpha \in \mathbb{Q}^* \text{ et } x \text{ is root of the equation } x(x-1) = \alpha^{11} \right\}$$

$$S_1 = \left\{ \left(x, (\alpha x(x-1)^3)^{\frac{1}{3}} \right) \mid \alpha \in \mathbb{Q}^* \text{ et } x \text{ is root of the equation } x(x-1) = \frac{1}{\alpha^{11}} \right\}$$

2. Auxiliary Results

For a divisor D on C , we note $\mathcal{L}(D)$ the $\bar{\mathbb{Q}}$ -vector space of rational functions f defined on \mathbb{Q} such that $f = 0$ or $\text{div}(f) \geq -D$; $l(D)$ denotes the $\bar{\mathbb{Q}}$ -dimension of $\mathcal{L}(D)$.

Lemma 2.1 we have: $J_{4,4}(11)(\mathbb{Q}) \cong \mathbb{Z}/11\mathbb{Z}$

Proof: According to Gross and Rohrlich ([6], p. 219), we have: $J_{4,4}(11)(\mathbb{Q})_{\text{torsion}} \cong \mathbb{Z}/11\mathbb{Z}$, and According Faddeev [5], on a: $J_{4,4}(11)(\mathbb{Q})_{\text{torsion}} \cong J_{4,4}(11)(\mathbb{Q})$.

Lemma 2.2 $\mathcal{C}_{4,4}(11) : y^{11} = x^4(x-1)^4$, we have:

(i)

$$\text{div}(x) = 11P_0 - 11P_\infty;$$

$$\text{div}(x-1) = 11P_1 - 11P_\infty;$$

$$\text{div}(y) = 4P_0 + 4P_1 - 8P_\infty.$$

(ii)

$$\mathcal{L}(P_\infty) = \langle 1 \rangle,$$

$$\mathcal{L}(2P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)} \right\rangle = \mathcal{L}(3P_\infty),$$

$$\mathcal{L}(4P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5} \right\rangle = \mathcal{L}(5P_\infty),$$

$$\mathcal{L}(6P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2} \right\rangle = \mathcal{L}(7P_\infty),$$

$$\mathcal{L}(8P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y \right\rangle = \mathcal{L}(9P_\infty),$$

$$\mathcal{L}(10P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y, \frac{x^3(x-1)^3}{y^7} \right\rangle$$

$$\mathcal{L}(11P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y, \frac{x^3(x-1)^3}{y^7}, x \right\rangle$$

$$\mathcal{L}(12P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y, \frac{x^3(x-1)^3}{y^7}, x, \frac{y^7}{x^2(x-1)^2} \right\rangle$$

Proof: Let x, y be the rational functions on \mathbb{Q} given by: $x(X, Y, Z) = \frac{X}{Z}$ and $y(X, Y, Z) = \frac{Y}{Z}$, which allows to give the projective form of the curve

$$C_{4,4}(11) : Y^{11} = X^4 Z^7 (X - Z)^4. (i)$$

(i) A)

$$div(x) = div\left(\frac{X}{Z}\right) = (X = 0) \cdot C_{4,4}(11) - (Z = 0) \cdot C_{4,4}(11)$$

a) For $X = 0$, we have $Y^{11} = 0$; pour $Z = 1$ we obtain the point $P_0 = (0, 0, 1)$ with an order of multiplicity equal to 11. Hence

$$(X = 0) \cdot C_{4,4}(11) = 11(P_0). \tag{1}$$

b) The same goes for $Z = 0$, we have $Y^{11} = 0$;for $X = 1$ we obtain the point $P_\infty = (1, 0, 0)$ with an order of multiplicity equal to 11. Hence

$$(Z = 0) \cdot C_{4,4}(11) = 11(P_\infty). \tag{2}$$

The relations (1) and (2) give

$$div(x) = 11(P_0) - 11(P_\infty).$$

B)

$$div(x-1) = div\left(\frac{X-Z}{Z}\right) = (X = Z) \cdot C_{4,4}(11) - (Z = 0) \cdot C_{4,4}(11)$$

a) For $X = Z$, the relation (i) give $Y^{11} = 0$.
We thus obtain the point $P_1 = (1, 0, 1)$ with an order multiplicity equal to 11. Hence

$$(X = Z) \cdot C_{4,4}(11) = 11(P_1). \tag{3}$$

b) For $Z = 0$, we have $Y^{11} = 0$; for $X = 1$ We thus obtain the point $P_\infty = (1, 0, 0)$ with an order multiplicity equal to 11. Hence

$$(Z = 0) \cdot C_{4,4}(11) = 11(P_\infty). \tag{4}$$

From relations 3 and 4 we deduce

$$\operatorname{div}(x-1) = 11(P_0) - 11(P_\infty).$$

C)

$$\operatorname{div}(y) = \operatorname{div}\left(\frac{Y}{Z}\right) = (Y=0) \cdot \mathcal{C}_{4,4} - (Z=0) \cdot \mathcal{C}_{4,4}.$$

a) For $Y=0$, we have $X^4(X-1)^4 = 0$ when $Z=1$; this give $X^4 = 0$ or $(X-1)^4 = 0$. Hence

$$(Y=0) \cdot \mathcal{C}_{4,4}(11) = 4(P_0) + 4(P_1). \quad (5)$$

b) the equation (i) can be written as: $Y^8 = X^4Y^{-3}Z^7(X-Z)^4$
Thus $Z=0$ we $X=1$, we have $Y^8 = 0$ we obtain the point $P_\infty = (1, 0, 0)$ with an order multiplicity equal to 8. Hence

$$(Z=0) \cdot \mathcal{C}_{4,4} = 8(P_\infty) \quad (6)$$

The relations (5) et (6) lead to the fact that $\operatorname{div}(y) = 4(P_0) + 4(P_1) - 8(P_\infty)$.

(ii) Resulte of (i).

Corollary 2.1 The following results are consequences of Lemma 2.

$$1) \quad 11j(P_0) = 11j(P_1) = 0$$

$$2) \quad 4j(P_0) = -4j(P_1)$$

So $j(P_0)$ and $j(P_1)$ generate the same group $J_{4,4}(11)(\mathbb{Q})$ isomorphic to $\mathbb{Z}/11\mathbb{Z}$.

Thus we have $J_{4,4}(11)(\mathbb{Q}) \cong \mathbb{Z}/11\mathbb{Z} = \{mj(P_0), 0 \leq m \leq 10\}$.

3. Demonstration of the Theorem

3.1. Quadratic Points on $\mathcal{C}_{4,4}(11)$

Let $R \in \mathcal{C}_{4,4}(11)(\mathbb{Q})$ with $[\mathbb{Q}(R) : \mathbb{Q}] = 2$. Let R_1 and R_2 be the conjugates of R in the Galois sense, and work with $t = [R_1 + R_2 - 2P_\infty]$ which is a point of $J_{4,4}(11)(\mathbb{Q}) = \{mj(P_0), 0 \leq m \leq 10\}$; so $t = mj(P_0)$ with $0 \leq m \leq 10$, thus

$$[R_1 + R_2 - 2P_\infty] = mj(P_0) \text{ with } 0 \leq m \leq 10 \quad (k)$$

We notice that $R \notin \{P_0, P_1, P_\infty\}$.

1st case $m = 0$.

The formula (k) becomes $[R_1 + R_2 - 2P_\infty] = 0$.

There is a rational function f such that $\operatorname{div}(f) = R_1 + R_2 - 2P_\infty$, so $f \in \mathcal{L}(2P_\infty)$. According to the lemma

2, we have $f = a + b\frac{y^3}{x(x-1)}$ with $a \neq 0$ (otherwise one of R_i would be equal to P_0) and $b \neq 0$ (otherwise

$\mathcal{L}(2P_\infty) = \mathcal{L}(P_\infty)$).

At points R_i we have:

$$a + b\frac{y^3}{x(x-1)} = 0$$

$$\iff y^3 = -\frac{a}{b}x(x-1)$$

$$\iff y = \left(-\frac{a}{b}x(x-1)\right)^{\frac{1}{3}}$$

On the other hand, we have:

$$\begin{aligned}
 y^{11} &= x^4(x - 1)^4 \\
 \iff \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x - 1))^{\frac{11}{3}} &= x^4(x - 1)^4 \\
 \iff \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x - 1))^4 (x(x - 1))^{-\frac{1}{3}} &= x^4(x - 1)^4 \\
 \iff \left(-\frac{a}{b}\right)^{\frac{11}{3}} x^4(x - 1)^4 (x(x - 1))^{-\frac{1}{3}} &= x^4(x - 1)^4 \\
 \iff \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x - 1))^{-\frac{1}{3}} &= 1 \\
 \iff \left(-\frac{a}{b}\right)^{11} (x(x - 1))^{-1} &= 1 \\
 \iff (x(x - 1)) &= \left(-\frac{a}{b}\right)^{11} \\
 \iff (x(x - 1)) &= \alpha^{11}
 \end{aligned}$$

We thus find a family of points:

$$\mathcal{S}_0 = \left\{ \left(x, (\alpha x(x - 1))^{\frac{1}{3}}\right) \mid \alpha \in \mathbb{Q}^* \text{ and } x \text{ is the root of the equation } x(x - 1) = \alpha^{11} \right\}$$

2nd case $m = 1$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = j(P_0) = (4 - 3)j(P_0) = 4j(P_0) - 3j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned}
 [R_1 + R_2 - 2P_\infty] &= 4j(P_0) - 3j(P_0) = -4j(P_1) - 3j(P_0) \\
 &= [-4P_1 + 4P_\infty - 3P_0 + 3P_\infty] = [-3P_0 - 4P_1 + 7P_\infty]
 \end{aligned}$$

Hence

$$[R_1 + R_2 + 3P_0 + 4P_1 - 9P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + 3P_0 + 4P_1 - 9P_\infty, \text{ so } f \in \mathcal{L}(9P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have $\text{ord}_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = ey$ we have $e \neq 0$ otherwise $\text{ord}_{P_1} f \neq 4$ which is absurd. At points R_i , we have $0 = ey$ hence $y = 0$ and therefore $0 = x^4(x-1)^4$ either $x = 0$ or $(x-1)^4 = 0$ we find the points P_0 and P_1 which is absurd.

3rd case $m = 2$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 2j(P_0) = (4-2)j(P_0) = 4j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 - 2P_\infty] &= 4j(P_0) - 2j(P_0) = -4j(P_1) - 2j(P_0) \\ &= [-4P_1 + 4P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 4P_1 + 6P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + 2P_0 + 4P_1 - 8P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + 2P_0 + 4P_1 - 8P_\infty, \text{ so } f \in \mathcal{L}(8P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have $\text{ord}_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = ey$ we have $e \neq 0$ otherwise $\text{ord}_{P_1} f \neq 4$ wich this absurd. At points R_i , we have $0 = ey$ hence $y = 0$ and therefore $0 = x^4(x-1)^4$ either $x = 0$ or $(x-1)^4 = 0$ we find the points P_0 and P_1 which is absurd.

4th case $m = 3$.

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 3j(P_0) = (4-1)j(P_0) = 4j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) - j(P_0) = -4j(P_1) - j(P_0) = [-4P_1 + 4P_\infty - P_0 + P_\infty] = [-P_0 - 4P_1 + 5P_\infty]$$

Hence

$$[R_1 + R_2 + P_0 + 4P_1 - 7P_\infty] = 0$$

There exists a rational function

$$f \text{ such that } \text{div}(f) = R_1 + R_2 + P_0 + 4P_1 - 7P_\infty, \text{ so } f \in \mathcal{L}(7P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have $ord_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = 0$ which is absurd.

5th case $m = 4$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) = -4j(P_1) = [-4P_1 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + 4P_1 - 6P_\infty] = 0$$

There exists a rational function f such that $div(f) = R_1 + R_2 + 4P_1 - 6P_\infty$, so $f \in \mathcal{L}(6P_\infty)$.

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have $ord_{P_1} f = 4$ hence $a = b = c = d = 0$ and therefore $f = 0$ which is absurd.

6th case $m = 5$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 5j(P_0) = (11 - 6)j(P_0) = 11j(P_0) - 6j(P_0)$$

At points R_i , we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 6j(P_0) = -6j(P_0) = [-6P_0 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + 6P_0 - 8P_\infty] = 0$$

There exists a rational function f such that $div(f) = R_1 + R_2 + 6P_0 - 8P_\infty$, so $f \in \mathcal{L}(8P_\infty)$.

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have $ord_{P_0} f = 6$ hence

$a = b = c = d = e = 0$ so $f = 0$ which contradicts the fact that $ord_{P_0} f = 6$ absurd.

7th case $m = 6$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 6j(P_0) = (8 - 2)j(P_0) = 8j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 - 2P_\infty] &= 8j(P_0) - 2j(P_0) = -8j(P_1) - 2j(P_0) \\ &= [-8P_1 + 8P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 8P_1 + 10P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + 2P_0 + 8P_1 - 12P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + 2P_0 + 8P_1 - 12P_\infty, \text{ so } f \in \mathcal{L}(12P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7} + e_2x + e_3 \frac{y^7}{x^2(x-1)^2}$$

We have $\text{ord}_{P_1} f = 8$, hence $a + e_2 = 0$ and $b = c = d = e = e_1 = e_3 = 0$; so $f = e_2(x-1)$ and therefore one of R_i should be equal to P_1 which is absurd.

8th case $m = 7$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 7j(P_0) = (11-4)j(P_0) = 11j(P_0) - 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 4j(P_0) = -4j(P_0) = [-4P_0 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + 4P_0 - 6P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + 4P_0 - 6P_\infty, \text{ so } f \in \mathcal{L}(6P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have $\text{ord}_{P_0} f = 4$ hence $a = b = c = d = 0$ so $f = 0$ which is absurd.

9th case $m = 8$

The formula (k) becomes $[R_1 + R_2 - 2P_\infty] = 8j(P_0)$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 8j(P_0) = -8j(P_1) = [-8P_1 + 8P_\infty]$$

Hence

$$[R_1 + R_2 + 8P_1 - 10P_\infty] = 0$$

There exists a rational function f such that $div(f) = R_1 + R_2 + 8P_1 - 10P_\infty$, so $f \in \mathcal{L}(10P_\infty)$.

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7}$$

We have $ord_{P_1} f = 8$ hence

$a = b = c = d = e = e_1 = 0$ so $f = 0$, which is absurd.

10th case $m = 9$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 9j(P_0) = (11 - 2)j(P_0) = 11j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 2j(P_0) = -2j(P_0) = -2[P_0 - P_\infty] = [-2P_0 + 2P_\infty]$$

Hence

$$[R_1 + R_2 + 2P_0 - 4P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + 2P_0 - 4P_\infty, \text{ so } f \in \mathcal{L}(4P_\infty).$$

According to the lemma 2, we have

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$$

We have $ord_{P_0} f = 2$ hence $a = b = 0$ so $f = c \frac{x^2(x-1)^2}{y^5}$ we have $c \neq 0$ otherwise $ord_{P_0} f \neq 2$ which is absurd.

At points R_i , we have $0 = c \frac{x^2(x-1)^2}{y^5}$ hence $x^2(x-1)^2 = 0$ either $x = 0$ or $x = 1$ we find the points P_0 et P_1 which is absurd.

11th case $m = 10$

The formula (k) become

$$[R_1 + R_2 - 2P_\infty] = 10j(P_0) = (11 - 1)j(P_0) = 11j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - j(P_0) = -j(P_0) = -[P_0 - P_\infty] = [-P_0 + P_\infty]$$

Hence

$$[R_1 + R_2 + P_0 - 3P_\infty] = 0$$

There exists a rational function

$$f \text{ such that } \operatorname{div}(f) = R_1 + R_2 + P_0 - 3P_\infty,$$

So

$$f \in \mathcal{L}(3P_\infty) \text{ and as } \mathcal{L}(3P_\infty) = \mathcal{L}(2P_\infty)$$

So one of R_i should be equal to P_∞ which is absurd.

3.2. Cubic Points on $\mathcal{C}_{4,4}(11)$

Let $R \in \mathcal{C}_{4,4}(11)(\mathbb{Q})$ with $[\mathbb{Q}(R) : \mathbb{Q}] = 3$. Let R_1, R_2 and R_3 be the conjugates of R in the Galois sense, and work with $t = [R_1 + R_2 + R_3 - 3P_\infty]$ which is a point of $J_{4,4}(11)(\mathbb{Q}) = \{mj(P_0), 0 \leq m \leq 10\}$; so $t = mj(P_0)$ with $0 \leq m \leq 10$, thus

$$[R_1 + R_2 + R_3 - 3P_\infty] = mj(P_0) \text{ with } 0 \leq m \leq 10 \quad (t)$$

We notice that $R \notin \{P_0, P_1, P_\infty\}$.

1st case $m = 0$.

The formula (t) becomes $[R_1 + R_2 + R_3 - 3P_\infty] = 0$.

There exists a rational function f such that

$$\operatorname{div}(f) = R_1 + R_2 + R_3 - 3P_\infty, \text{ so } f \in \mathcal{L}(3P_\infty) \text{ and as } \mathcal{L}(3P_\infty) = \mathcal{L}(2P_\infty)$$

So one of the R_i should be equal to P_∞ which is absurd.

2nd case $m = 1$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = j(P_0) = (4-3)j(P_0) = 4j(P_0) - 3j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 + R_3 - 3P_\infty] &= 4j(P_0) - 3j(P_0) = -4j(P_1) - 3j(P_0) \\ &= [-4P_1 + 4P_\infty - 3P_0 + 3P_\infty] = [-3P_0 - 4P_1 + 7P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + R_3 + 3P_0 + 4P_1 - 10P_\infty] = 0$$

There exists a rational function f such that

$$\operatorname{div}(f) = R_1 + R_2 + R_3 + 3P_0 + 4P_1 - 10P_\infty, \text{ so } f \in \mathcal{L}(10P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7}$$

We have $\operatorname{ord}_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = ey + e_1 \frac{x^3(x-1)^3}{y^7}$; one of R_i should be equal to P_0 which

is absurd.

3rd case $m = 2$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 2j(P_0) = (4 - 2)j(P_0) = 4j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 - 2P_\infty] &= 4j(P_0) - 2j(P_0) = -4j(P_1) - 2j(P_0) \\ &= [-4P_1 + 4P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 4P_1 + 6P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + R_3 + 2P_0 + 4P_1 - 9P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 2P_0 + 4P_1 - 9P_\infty, \text{ so } f \in \mathcal{L}(9P_\infty) \text{ and as } \mathcal{L}(9P_\infty) = \mathcal{L}(8P_\infty)$$

Therefore one of R_i should be equal to P_∞ which is absurd.

4th case $m = 3$.

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 3j(P_0) = (4 - 1)j(P_0) = 4j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 + R_3 - 3P_\infty] &= 4j(P_0) - j(P_0) = -4j(P_1) - j(P_0) \\ &= [-4P_1 + 4P_\infty - P_0 + P_\infty] = [-P_0 - 4P_1 + 5P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + R_3 + P_0 + 4P_1 - 8P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + R_3 + P_0 + 4P_1 - 8P_\infty, \text{ so } f \in \mathcal{L}(8P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have $\text{ord}_{P_1} f = 4$ hence $a = b = c = d = 0$ so $f = ey$ we have $e \neq 0$ otherwise $\text{ord}_{P_1} f \neq 4$ which is absurd. At points R_i , we have $0 = ey$ hence $y = 0$ and therefore $0 = x^4(x-1)^4$ either $x = 0$ or $(x-1)^4 = 0$ we find the points P_0 and P_1 which is absurd.

5th case $m = 4$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 4j(P_0) = -4j(P_1) = [-4P_1 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 4P_1 - 7P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 4P_1 - 7P_\infty, \text{ so } f \in \mathcal{L}(7P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have $\text{ord}_{P_1} f = 4$ hence $a = b = c = d = 0$ and therefore one of $f = 0$ which is absurd.

6th case $m = 5$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 5j(P_0) = (11-6)j(P_0) = 11j(P_0) - 6j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 6j(P_0) = -6j(P_0) = [-6P_0 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 6P_0 - 9P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 6P_0 - 9P_\infty, \text{ so } f \in \mathcal{L}(9P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have $\text{ord}_{P_0} f = 6$ hence

$a = b = c = d = e = 0$ so $f = 0$ which contradicts the fact that $\text{ord}_{P_0} f = 6$ absurd.

7th case $m = 6$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 6j(P_0) = (11-5)j(P_0) = 11j(P_0) - 5j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 5j(P_0) = -5j(P_0) = [-5P_0 + 5P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 5P_0 - 8P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 5P_0 - 8P_\infty, \text{ so } f \in \mathcal{L}(8P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have $ord_{P_0} f = 5$, hence $a = b = c = d = e = 0$; so $f = 0$ which contradicts the fact that $ord_{P_0} f = 5$ absurd.
8th case $m = 7$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 7j(P_0) = (11 - 4)(jP_0) = 11j(P_0) - 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 4j(P_0) = -4j(P_0) = [-4P_0 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 4P_0 - 7P_\infty] = 0$$

There exists a rational function f such that

$$div(f) = R_1 + R_2 + R_3 + 4P_0 - 7P_\infty, \text{ so } f \in \mathcal{L}(7P_\infty) \text{ and as } \mathcal{L}(7P_\infty) = \mathcal{L}(6P_\infty)$$

So one of R_i should be equal to P_∞ which is absurd.

9th case $m = 8$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 8j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 8j(P_0) = -8j(P_1) = [-8P_1 + 8P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 8P_1 - 11P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 8P_1 - 11P_\infty, \text{ so } f \in \mathcal{L}(11P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7} + e_2x$$

We have $\text{ord}_{P_1} f = 8$, we must have $a + e_2 = 0$ and $b = c = d = e = e_1 = 0$; so $f = e_2(x-1)$ and therefore one of R_i should be equal to P_1 which is absurd.

10th case $m = 9$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 9j(P_0) = (11-2)(jP_0) = 11j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 2j(P_0) = -2j(P_0) = -2[P_0 - P_\infty] = [-2P_0 + 2P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 2P_0 - 5P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 2P_0 - 5P_\infty, \text{ so } f \in \mathcal{L}(5P_\infty)$$

According to the lemma 2, we have $f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$

We have $\text{ord}_{P_0} f = 2$ hence $a = b = 0$ so $f = c \frac{x^2(x-1)^2}{y^5}$ we have $c \neq 0$ otherwise $\text{ord}_{P_0} f \neq 2$ ce qui est absurde. At points R_i , we have $0 = c \frac{x^2(x-1)^2}{y^5}$ hence $x^2(x-1)^2 = 0$ either $x = 0$ or $x = 1$ we find the points P_0 and P_1 which is absurd.

11th case $m = 10$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 10j(P_0) = (11-1)j(P_0) = 11j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - j(P_0) = -j(P_0) = [-P_0 + P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + P_0 - 4P_\infty] = 0$$

There exists a rational function f such that

$$\text{div}(f) = R_1 + R_2 + R_3 + P_0 - 4P_\infty, \text{ so } f \in \mathcal{L}(4P_\infty)$$

And therefore

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$$

$$\text{ord}_{P_0} f = 1 \implies a = 0; \text{ so } f = b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$$

$b \neq 0$ otherwise $\text{ord}_{P_0} f \neq 1$; which is absurd.

$c \neq 0$ sinon $f \in \mathcal{L}(3P_\infty)$; which is absurd.

At points R_i , we have

$$0 = b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} :$$

$$b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} = 0$$

$$\iff y^8 = -\frac{c}{b} x^3(x-1)^3$$

$$\iff y = \left(-\frac{c}{b} (x(x-1))^3 \right)^{\frac{1}{8}}$$

On the other hand, we have:

$$y^{11} = x^4(x-1)^4$$

$$\iff \left(-\frac{c}{b} (x(x-1))^3 \right)^{\frac{11}{8}} = x^4(x-1)^4$$

$$\iff \left(-\frac{c}{b} \right)^{\frac{11}{8}} (x(x-1))^{\frac{33}{8}} = x^4(x-1)^4$$

$$\iff \left(-\frac{c}{b} \right)^{\frac{11}{8}} (x(x-1))^{\frac{32}{8}} (x(x-1))^{\frac{1}{8}} = x^4(x-1)^4$$

$$\iff \left(-\frac{c}{b} \right)^{\frac{11}{8}} x^4(x-1)^4 (x(x-1))^{\frac{1}{8}} = x^4(x-1)^4$$

$$\iff \left(-\frac{c}{b} \right)^{\frac{11}{8}} (x(x-1))^{\frac{1}{8}} = 1$$

$$\iff \left(-\frac{c}{b} \right)^{11} x(x-1) = 1$$

$$\iff x(x-1) = \left(-\frac{b}{c}\right)^{11}$$

$$\iff x(x-1) = -\left(\frac{b}{c}\right)^{11}$$

We thus find a family of points:

$$S_1 = \left\{ \left(x, (\alpha(x(x-1))^3)^{\frac{1}{8}} \right) \mid \alpha \in \mathbb{Q}^* \text{ and } x \text{ is the root of the equation } x(x-1) = \frac{1}{\alpha^{11}} \right\}$$

4. Conclusion

Our note focuses on the determination of algebraic points on the curve $\mathcal{C}_{4,4}(11)$ of affine equation $y^{11} = x^4(x-1)^4$. The curve $\mathcal{C}_{4,4}(11)$ is a special case of the quotients of Fermat curves. In this note we have explicitly determined the algebraic points of degree at most 3 on the curve $\mathcal{C}_{4,4}(11)$ on \mathbb{Q} . To do this we determined the quadratic points and the cubic points on $\mathcal{C}_{4,4}(11)$ on \mathbb{Q} .

It seems possible to determine explicitly the algebraic points of any given degree on the curve $\mathcal{C}_{4,4}(11)$.

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