

# Algebraic Points of Degree at Most 3 on the Affine Equation Curve $y^{11} = x^4(x - 1)^4$

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**Abstract:** The quotients of Fermat curves  $\mathcal{C}_{r,s}(p)$  are studied by Oumar SALL. Among these studies are the cases  $\mathcal{C}_{r,s}(11)$  for  $r = s = 1$ . Mamina COLY and Oumar SALL have explicitly determined the algebraic points of degree at most 3 on  $\mathbb{Q}$  for the cases  $\mathcal{C}_{r,s}(11)$  for  $r = s = 2$ . Our work focuses on determining explicitly the algebraic points of degree at most 3 on  $\mathbb{Q}$  on the curve  $\mathcal{C}_{4,4}(11)$  which is a special case of Fermat quotient curves. Our study concerns the cases  $\mathcal{C}_{r,s}(11)$  for  $r = s = 4$ . It seems that the finiteness of the Mordell-Weil group of rational points of the Jacobien  $J_{4,4}(11)(\mathbb{Q})$  is an essential condition. So to determine the algebraic points on the curve  $\mathcal{C}_{4,4}(11)$  we need a finiteness of the Mordell-Weil group of rational points of the Jacobien  $J_{4,4}(11)(\mathbb{Q})$ . The Mordell-Weil group  $J_{4,4}(11)(\mathbb{Q})$  of rational points of the Jacobien is finite according to Faddev. Our note is in this framework. Our essential tools in this note are the Mordell-Weil group  $J_{4,4}(11)(\mathbb{Q})$  of the Jacobien of  $\mathcal{C}_{4,4}(11)$  the Abel-Jacobi theorem and the study of linear systems on the curve  $\mathcal{C}_{4,4}(11)$ . The result obtained concerns some quotients of Fermat curves. Indeed, the curve  $\mathcal{C}_{4,4}(11)$  which is the subject of our study, the set of algebraic points of degree at most 3 on  $\mathbb{Q}$  has been determined in an explicit way, to achieve this we have determined the quadratic points on the curve  $\mathcal{C}_{4,4}(11)$  on  $\mathbb{Q}$  and the cubic points on the curve  $\mathcal{C}_{4,4}(11)$  on  $\mathbb{Q}$ .

**Keywords:** Planes Curves, Degree of Algebraic Points, Jacobien

## 1. Introduction

Let  $\mathcal{C}$  be a smooth algebraic curve defined on  $\mathbb{Q}$ . Let  $K$  be a field of numbers we note  $\mathcal{C}(K)$  the set of points on  $\mathcal{C}$  with coordinates in  $K$ , and  $\bigcup_{[K:\mathbb{Q}] \leq d} \mathcal{C}(K)$  the set of points on  $\mathcal{C}$  with coordinates in  $K$  of degree at most  $d$  on  $\mathbb{Q}$ . The degree of a point  $R$  of  $\mathcal{C}$  algebraic on  $\mathbb{Q}$  is defined as the degree of its defining field on  $\mathbb{Q}$ ; in other words  $\deg(R) = [\mathbb{Q}(R) : \mathbb{Q}]$ .

In this note we will focus on the curve  $\mathcal{C}_{4,4}(11)$  with affine equation  $y^{11} = x^4(x - 1)^4$  which is a special case of quotients of Fermat curves  $\mathcal{C}_{r,s}(p) : y^p = x^r(x - 1)^s$ ,  $1 \leq r, s$ ;  $r + s \leq p - 1$  studied in [8-10]. The cases  $\mathcal{C}_{r,s}(11)$  for  $r = s = 2$  are studied

in [2]. See [1, 3, 4, 11] for other explicit examples. Indeed,  $\mathcal{C}$  corresponds to the curve  $\mathcal{C}_{4,4}(11)$ . The curves  $\mathcal{C}_{r,s}(p)$  are quotients of  $F_p$  [6, 15].

We denote by  $J_{4,4}(11)$  the Jacobien of  $\mathcal{C}_{4,4}(11)$  and by  $j(P)$  the class denoted  $[P - P_\infty]$  of  $P - P_\infty$ , that is to say  $j$  is the Jacobien fold  $\mathcal{C}_{4,4}(11) \rightarrow J_{4,4}(11)$ . The Mordell-Weil group  $J_{4,4}(11)(\mathbb{Q})$  of the rational points of the Jacobien is finite [5, 6, 12, 13]. The curve  $\mathcal{C}_{4,4}(11)$  in projective is  $\mathcal{C}_{4,4}(11) : Y^{11} = X^4 Z^7 (X - Z)^4$ . Let us note  $P_0, P_1$  and  $P_\infty$  the points defined by:  $P_0 = (0, 0, 1)$ ;  $P_1 = (1, 0, 1)$  and  $P_\infty = (1, 0, 0)$ . It follows from the work of Gross-Rohrlich in [6], that

$$\bigcup_{[K:\mathbb{Q}] \leq 2} \mathcal{C}_{1,1}(11)(K) = \left\{ \left( \frac{1}{2} \pm \sqrt{y^{11} + \frac{1}{4}}, y \right) \mid y \in \mathbb{Q} \right\} \cup \{ P_\infty \}$$

In this note we determine the set

$$\bigcup_{[K:\mathbb{Q}] \leq 3} \mathcal{C}_{4,4}(11)(K)$$

Our main result is the following: *Theorem 1.1* The set of algebraic points of degree at most 3 on  $\mathbb{Q}$  of the curve  $\mathcal{C}_{4,4}(11)$  is given by:

$$\bigcup_{[K:\mathbb{Q}] \leq 3} \mathcal{C}_{4,4}(11)(K) = \{P_0, P_1, P_\infty\} \cup S_0 \cup S_1$$

with

$$S_0 = \left\{ \left( x, (\alpha x(x-1))^{\frac{1}{3}} \right) \mid \alpha \in \mathbb{Q}^* \text{ et } x \text{ is root of the equation } x(x-1) = \alpha^{11} \right\}$$

$$S_1 = \left\{ \left( x, (\alpha x(x-1)^3)^{\frac{1}{8}} \right) \mid \alpha \in \mathbb{Q}^* \text{ et } x \text{ is root of the equation } x(x-1) = \frac{1}{\alpha^{11}} \right\}$$

## 2. Auxiliary Results

For a divisor  $D$  on  $C$ , we note  $\mathcal{L}(D)$  the  $\bar{\mathbb{Q}}$ -vector space of rational functions  $f$  defined on  $\mathbb{Q}$  such that  $f = 0$  or  $\text{div}(f) \geq -D$ ;  $l(D)$  denotes the  $\bar{\mathbb{Q}}$ -dimension of  $\mathcal{L}(D)$ .

*Lemma 2.1* we have:  $J_{4,4}(11)(\mathbb{Q}) \cong \mathbb{Z}/11\mathbb{Z}$

*Proof:* According to Gross and Rohrlich ([6], p. 219), we have:  $J_{4,4}(11)(\mathbb{Q})_{\text{torsion}} \cong \mathbb{Z}/11\mathbb{Z}$ , and According Faddeev [5], on a:  $J_{4,4}(11)(\mathbb{Q})_{\text{torsion}} \cong J_{4,4}(11)(\mathbb{Q})$ .

*Lemma 2.2*  $\mathcal{C}_{4,4}(11) : y^{11} = x^4(x-1)^4$ , we have:

(i)

$$\text{div}(x) = 11P_0 - 11P_\infty;$$

$$\text{div}(x-1) = 11P_1 - 11P_\infty;$$

$$\text{div}(y) = 4P_0 + 4P_1 - 8P_\infty.$$

(ii)

$$\mathcal{L}(P_\infty) = \langle 1 \rangle,$$

$$\mathcal{L}(2P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)} \right\rangle = \mathcal{L}(3P_\infty),$$

$$\mathcal{L}(4P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5} \right\rangle = \mathcal{L}(5P_\infty),$$

$$\mathcal{L}(6P_\infty) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2} \right\rangle = \mathcal{L}(7P_\infty),$$

$$\mathcal{L}(8P_{\infty}) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y \right\rangle = \mathcal{L}(9P_{\infty}),$$

$$\mathcal{L}(10P_{\infty}) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y, \frac{x^3(x-1)^3}{y^7} \right\rangle$$

$$\mathcal{L}(11P_{\infty}) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y, \frac{x^3(x-1)^3}{y^7}, x \right\rangle$$

$$\mathcal{L}(12P_{\infty}) = \left\langle 1, \frac{y^3}{x(x-1)}, \frac{x^2(x-1)^2}{y^5}, \frac{x(x-1)}{y^2}, y, \frac{x^3(x-1)^3}{y^7}, x, \frac{y^7}{x^2(x-1)^2} \right\rangle$$

*Proof:* Let  $x, y$  be the rational functions on  $\mathbb{Q}$  given by:  $x(X, Y, Z) = \frac{X}{Z}$  and  $y(X, Y, Z) = \frac{Y}{Z}$ , which allows to give the projective form of the curve

$$\mathcal{C}_{4,4}(11) : Y^{11} = X^4 Z^7 (X - Z)^4. (i)$$

(i) A)

$$\text{div}(x) = \text{div}\left(\frac{X}{Z}\right) = (X = 0) \cdot \mathcal{C}_{4,4}(11) - (Z = 0) \cdot \mathcal{C}_{4,4}(11)$$

a) For  $X = 0$ , we have  $Y^{11} = 0$ ; pour  $Z = 1$  we obtain the point  $P_0 = (0, 0, 1)$  with an order of multiplicity equal to 11. Hence

$$(X = 0) \cdot \mathcal{C}_{4,4}(11) = 11(P_0). \quad (1)$$

b) The same goes for  $Z = 0$ , we have  $Y^{11} = 0$ ; for  $X = 1$  we obtain the point  $P_{\infty} = (1, 0, 0)$  with an order of multiplicity equal to 11. Hence

$$(Z = 0) \cdot \mathcal{C}_{4,4}(11) = 11(P_{\infty}). \quad (2)$$

The relations (1) and (2) give

$$\text{div}(x) = 11(P_0) - 11(P_{\infty}).$$

B)

$$\text{div}(x-1) = \text{div}\left(\frac{X-Z}{Z}\right) = (X = Z) \cdot \mathcal{C}_{4,4}(11) - (Z = 0) \cdot \mathcal{C}_{4,4}(11)$$

a) For  $X = Z$ , the relation (i) give  $Y^{11} = 0$ .

We thus obtain the point  $P_1 = (1, 0, 1)$  with an order multiplicity equal to 11. Hence

$$(X = Z) \cdot \mathcal{C}_{4,4}(11) = 11(P_1). \quad (3)$$

b) For  $Z = 0$ , we have  $Y^{11} = 0$ ; for  $X = 1$  We thus obtain the point  $P_{\infty} = (1, 0, 0)$  with an order multiplicity equal to 11. Hence

$$(Z = 0) \cdot \mathcal{C}_{4,4}(11) = 11(P_{\infty}). \quad (4)$$

From relations 3 and 4 we deduce

$$\operatorname{div}(x-1) = 11(P_0) - 11(P_\infty).$$

C)

$$\operatorname{div}(y) = \operatorname{div}\left(\frac{Y}{Z}\right) = (Y = 0) \cdot \mathcal{C}_{4,4} - (Z = 0) \cdot \mathcal{C}_{4,4}.$$

a) For  $Y = 0$ , we have  $X^4(X-1)^4 = 0$  when  $Z = 1$ ; this give  $X^4 = 0$  or  $(X-1)^4 = 0$ .  
Hence

$$(Y = 0) \cdot \mathcal{C}_{4,4}(11) = 4(P_0) + 4(P_1). \quad (5)$$

b) the equation (i) can be written as:  $Y^8 = X^4 Y^{-3} Z^7 (X - Z)^4$   
Thus  $Z = 0$  we  $X = 1$ , we have  $Y^8 = 0$  we obtain the point  $P_\infty = (1, 0, 0)$  with an order multiplicity equal to 8. Hence

$$(Z = 0) \cdot \mathcal{C}_{4,4} = 8(P_\infty) \quad (6)$$

The relations (5) et (6) lead to the fact that  $\operatorname{div}(y) = 4(P_0) + 4(P_1) - 8(P_\infty)$ .

(ii) Resulte of (i).

*Corollary 2.1* The following results are consequences of Lemma 2.

$$1) \quad 11j(P_0) = 11j(P_1) = 0$$

$$2) \quad 4j(P_0) = -4j(P_1)$$

So  $j(P_0)$  and  $j(P_1)$  generate the same group  $J_{4,4}(11)(\mathbb{Q})$  isomorphic to  $\mathbb{Z}/11\mathbb{Z}$ .

Thus we have  $J_{4,4}(11)(\mathbb{Q}) \cong \mathbb{Z}/11\mathbb{Z} = \{mj(P_0), 0 \leq m \leq 10\}$ .

### 3. Demonstration of the Theorem

#### 3.1. Quadratic Points on $\mathcal{C}_{4,4}(11)$

Let  $R \in \mathcal{C}_{4,4}(11)(\mathbb{Q})$  with  $[\mathbb{Q}(R) : \mathbb{Q}] = 2$ . Let  $R_1$  and  $R_2$  be the conjugates of  $R$  in the Galois sense, and work with  $t = [R_1 + R_2 - 2P_\infty]$  which is a point of  $J_{4,4}(11)(\mathbb{Q}) = \{mj(P_0), 0 \leq m \leq 10\}$ ; so  $t = mj(P_0)$  with  $0 \leq m \leq 10$ , thus

$$[R_1 + R_2 - 2P_\infty] = mj(P_0) \text{ with } 0 \leq m \leq 10 \quad (k)$$

We notice that  $R \notin \{P_0, P_1, P_\infty\}$ .

1<sup>st</sup> case  $m = 0$ .

The formula (k) becomes  $[R_1 + R_2 - 2P_\infty] = 0$ .

There is a rational function  $f$  such that  $\operatorname{div}(f) = R_1 + R_2 - 2P_\infty$ , so  $f \in \mathcal{L}(2P_\infty)$ . According to the lemma 2, we have  $f = a + b \frac{y^3}{x(x-1)}$  with  $a \neq 0$  (otherwise one of  $R_i$  would be equal to  $P_0$ ) and  $b \neq 0$  (otherwise

$\mathcal{L}(2P_\infty) = \mathcal{L}(P_\infty)$ ).

At points  $R_i$  we have:

$$a + b \frac{y^3}{x(x-1)} = 0$$

$$\Longleftrightarrow y^3 = -\frac{a}{b}x(x-1)$$

$$\Longleftrightarrow y = \left(-\frac{a}{b}x(x-1)\right)^{\frac{1}{3}}$$

On the other hand, we have:

$$\begin{aligned}
 y^{11} &= x^4(x-1)^4 \\
 \Leftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x-1))^{\frac{11}{3}} &= x^4(x-1)^4 \\
 \Leftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x-1))^4 (x(x-1))^{\frac{-1}{3}} &= x^4(x-1)^4 \\
 \Leftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} x^4(x-1)^4 (x(x-1))^{\frac{-1}{3}} &= x^4(x-1)^4 \\
 \Leftrightarrow \left(-\frac{a}{b}\right)^{\frac{11}{3}} (x(x-1))^{\frac{-1}{3}} &= 1 \\
 \Leftrightarrow \left(-\frac{a}{b}\right)^{11} (x(x-1))^{-1} &= 1 \\
 \Leftrightarrow (x(x-1)) &= \left(-\frac{a}{b}\right)^{11} \\
 \Leftrightarrow (x(x-1)) &= \alpha^{11}
 \end{aligned}$$

We thus find a family of points:

$$\mathcal{S}_0 = \left\{ \left( x, (\alpha x(x-1))^{\frac{1}{3}} \right) \mid \alpha \in \mathbb{Q}^* \text{ and } x \text{ is the root of the equation } x(x-1) = \alpha^{11} \right\}$$

2<sup>nd</sup> case  $m = 1$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = j(P_0) = (4-3)j(P_0) = 4j(P_0) - 3j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned}
 [R_1 + R_2 - 2P_\infty] &= 4j(P_0) - 3j(P_0) = -4j(P_1) - 3j(P_0) \\
 &= [-4P_1 + 4P_\infty - 3P_0 + 3P_\infty] = [-3P_0 - 4P_1 + 7P_\infty]
 \end{aligned}$$

Hence

$$[R_1 + R_2 + 3P_0 + 4P_1 - 9P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + 3P_0 + 4P_1 - 9P_\infty, \text{ so } f \in \mathcal{L}(9P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have  $\text{ord}_{P_1} f = 4$  hence  $a = b = c = d = 0$  so  $f = ey$  we have  $e \neq 0$  otherwise  $\text{ord}_{P_1} f \neq 4$  which is absurd. At points  $R_i$ , we have  $0 = ey$  hence  $y = 0$  and therefore  $0 = x^4(x-1)^4$  either  $x = 0$  or  $(x-1)^4 = 0$  we find the points  $P_0$  and  $P_1$  which is absurd.

3<sup>rd</sup> case  $m = 2$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 2j(P_0) = (4-2)j(P_0) = 4j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) - 2j(P_0) = -4j(P_1) - 2j(P_0)$$

$$= [-4P_1 + 4P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 4P_1 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + 2P_0 + 4P_1 - 8P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + 2P_0 + 4P_1 - 8P_\infty, \text{ so } f \in \mathcal{L}(8P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have  $\text{ord}_{P_1} f = 4$  hence  $a = b = c = d = 0$  so  $f = ey$  we have  $e \neq 0$  otherwise  $\text{ord}_{P_1} f \neq 4$  which is absurd. At points  $R_i$ , we have  $0 = ey$  hence  $y = 0$  and therefore  $0 = x^4(x-1)^4$  either  $x = 0$  or  $(x-1)^4 = 0$  we find the points  $P_0$  and  $P_1$  which is absurd.

4<sup>th</sup> case  $m = 3$ .

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 3j(P_0) = (4-1)j(P_0) = 4j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) - j(P_0) = -4j(P_1) - j(P_0) = [-4P_1 + 4P_\infty - P_0 + P_\infty] = [-P_0 - 4P_1 + 5P_\infty]$$

Hence

$$[R_1 + R_2 + P_0 + 4P_1 - 7P_\infty] = 0$$

There exists a rational function

$$f \text{ such that } \text{div}(f) = R_1 + R_2 + P_0 + 4P_1 - 7P_\infty, \text{ so } f \in \mathcal{L}(7P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have  $\text{ord}_{P_1} f = 4$  hence  $a = b = c = d = 0$  so  $f = 0$  which is absurd.

5<sup>th</sup> case  $m = 4$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 4j(P_0) = -4j(P_1) = [-4P_1 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + 4P_1 - 6P_\infty] = 0$$

There exists a rational function  $f$  such that  $\text{div}(f) = R_1 + R_2 + 4P_1 - 6P_\infty$ , so  $f \in \mathcal{L}(6P_\infty)$ .

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have  $\text{ord}_{P_1} f = 4$  hence  $a = b = c = d = 0$  and therefore  $f = 0$  which is absurd.

6<sup>th</sup> case  $m = 5$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 5j(P_0) = (11-6)j(P_0) = 11j(P_0) - 6j(P_0)$$

At points  $R_i$ , we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 6j(P_0) = -6j(P_0) = [-6P_0 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + 6P_0 - 8P_\infty] = 0$$

There exists a rational function  $f$  such that  $\text{div}(f) = R_1 + R_2 + 6P_0 - 8P_\infty$ , so  $f \in \mathcal{L}(8P_\infty)$ .

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have  $\text{ord}_{P_0} f = 6$  hence

$a = b = c = d = e = 0$  so  $f = 0$  which contradicts the fact that  $\text{ord}_{P_0} f = 6$  absurd.

7<sup>th</sup> case  $m = 6$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 6j(P_0) = (8-2)j(P_0) = 8j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 - 2P_\infty] &= 8j(P_0) - 2j(P_0) = -8j(P_1) - 2j(P_0) \\ &= [-8P_1 + 8P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 8P_1 + 10P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + 2P_0 + 8P_1 - 12P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + 2P_0 + 8P_1 - 12P_\infty, \text{ so } f \in \mathcal{L}(12P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7} + e_2x + e_3 \frac{y^7}{x^2(x-1)^2}$$

We have  $\text{ord}_{P_1} f = 8$ , hence  $a + e_2 = 0$  and  $b = c = d = e = e_1 = e_3 = 0$ ; so  $f = e_2(x-1)$  and therefore one of  $R_i$  should be equal to  $P_1$  which is absurd.

8<sup>th</sup> case  $m = 7$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 7j(P_0) = (11-4)j(P_0) = 11j(P_0) - 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 4j(P_0) = -4j(P_0) = [-4P_0 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + 4P_0 - 6P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + 4P_0 - 6P_\infty, \text{ so } f \in \mathcal{L}(6P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have  $\text{ord}_{P_0} f = 4$  hence  $a = b = c = d = 0$  so  $f = 0$  which is absurd.

9<sup>th</sup> case  $m = 8$

The formula (k) becomes  $[R_1 + R_2 - 2P_\infty] = 8j(P_0)$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 8j(P_0) = -8j(P_1) = [-8P_1 + 8P_\infty]$$



Hence

$$[R_1 + R_2 + 8P_1 - 10P_\infty] = 0$$

There exists a rational function  $f$  such that  $\text{div}(f) = R_1 + R_2 + 8P_1 - 10P_\infty$ , so  $f \in \mathcal{L}(10P_\infty)$ .

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7}$$

We have  $\text{ord}_{P_1} f = 8$  hence

$a = b = c = d = e = e_1 = 0$  so  $f = 0$ , which is absurd.

10<sup>th</sup> case  $m = 9$

The formula (k) becomes

$$[R_1 + R_2 - 2P_\infty] = 9j(P_0) = (11-2)j(P_0) = 11j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - 2j(P_0) = -2j(P_0) = -2[P_0 - P_\infty] = [-2P_0 + 2P_\infty]$$

Hence

$$[R_1 + R_2 + 2P_0 - 4P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + 2P_0 - 4P_\infty, \text{ so } f \in \mathcal{L}(4P_\infty).$$

According to the lemma 2, we have

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$$

We have  $\text{ord}_{P_0} f = 2$  hence  $a = b = 0$  so  $f = c \frac{x^2(x-1)^2}{y^5}$  we have  $c \neq 0$  otherwise  $\text{ord}_{P_0} f \neq 2$  which is absurd.

At points  $R_i$ , we have  $0 = c \frac{x^2(x-1)^2}{y^5}$  hence  $x^2(x-1)^2 = 0$  either  $x = 0$  or  $x = 1$  we find the points  $P_0$  et  $P_1$  which is absurd.

11<sup>th</sup> case  $m = 10$

The formula (k) become

$$[R_1 + R_2 - 2P_\infty] = 10j(P_0) = (11-1)j(P_0) = 11j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 - 2P_\infty] = 11j(P_0) - j(P_0) = -j(P_0) = -[P_0 - P_\infty] = [-P_0 + P_\infty]$$

Hence

$$[R_1 + R_2 + P_0 - 3P_\infty] = 0$$

There exists a rational function

$$f \text{ such that } \operatorname{div}(f) = R_1 + R_2 + P_0 - 3P_\infty,$$

So

$$f \in \mathcal{L}(3P_\infty) \text{ and as } \mathcal{L}(3P_\infty) = \mathcal{L}(2P_\infty)$$

So one of  $R_i$  should be equal to  $P_\infty$  which is absurd.

### 3.2. Cubic Points on $\mathcal{C}_{4,4}(11)$

Let  $R \in \mathcal{C}_{4,4}(11)(\mathbb{Q})$  with  $[\mathbb{Q}(R) : \mathbb{Q}] = 3$ . Let  $R_1, R_2$  and  $R_3$  be the conjugates of  $R$  in the Galois sense, and work with  $t = [R_1 + R_2 + R_3 - 3P_\infty]$  which is a point of  $J_{4,4}(11)(\mathbb{Q}) = \{mj(P_0), 0 \leq m \leq 10\}$ ; so  $t = mj(P_0)$  with  $0 \leq m \leq 10$ , thus

$$[R_1 + R_2 + R_3 - 3P_\infty] = mj(P_0) \text{ with } 0 \leq m \leq 10 \quad (t)$$

We notice that  $R \notin \{P_0, P_1, P_\infty\}$ .

1<sup>st</sup> case  $m = 0$ .

The formula (t) becomes  $[R_1 + R_2 + R_3 - 3P_\infty] = 0$ .

There exists a rational function  $f$  such that

$$\operatorname{div}(f) = R_1 + R_2 + R_3 - 3P_\infty, \text{ so } f \in \mathcal{L}(3P_\infty) \text{ and as } \mathcal{L}(3P_\infty) = \mathcal{L}(2P_\infty)$$

So one of the  $R_i$  should be equal to  $P_\infty$  which is absurd.

2<sup>nd</sup> case  $m = 1$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = j(P_0) = (4-3)j(P_0) = 4j(P_0) - 3j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 + R_3 - 3P_\infty] &= 4j(P_0) - 3j(P_0) = -4j(P_1) - 3j(P_0) \\ &= [-4P_1 + 4P_\infty - 3P_0 + 3P_\infty] = [-3P_0 - 4P_1 + 7P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + R_3 + 3P_0 + 4P_1 - 10P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\operatorname{div}(f) = R_1 + R_2 + R_3 + 3P_0 + 4P_1 - 10P_\infty, \text{ so } f \in \mathcal{L}(10P_\infty).$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7}$$

We have  $\operatorname{ord}_{P_1} f = 4$  hence  $a = b = c = d = 0$  so  $f = ey + e_1 \frac{x^3(x-1)^3}{y^7}$ ; one of  $R_i$  should be equal to  $P_0$  which

is absurd.

3<sup>rd</sup> case  $m = 2$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 2j(P_0) = (4 - 2)j(P_0) = 4j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 - 2P_\infty] &= 4j(P_0) - 2j(P_0) = -4j(P_1) - 2j(P_0) \\ &= [-4P_1 + 4P_\infty - 2P_0 + 2P_\infty] = [-2P_0 - 4P_1 + 6P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + R_3 + 2P_0 + 4P_1 - 9P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 2P_0 + 4P_1 - 9P_\infty, \text{ so } f \in \mathcal{L}(9P_\infty) \text{ and as } \mathcal{L}(9P_\infty) = \mathcal{L}(8P_\infty)$$

Therefore one of  $R_i$  should be equal to  $P_\infty$  which is absurd.

4<sup>th</sup> case  $m = 3$ .

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 3j(P_0) = (4 - 1)j(P_0) = 4j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$\begin{aligned} [R_1 + R_2 + R_3 - 3P_\infty] &= 4j(P_0) - j(P_0) = -4j(P_1) - j(P_0) \\ &= [-4P_1 + 4P_\infty - P_0 + P_\infty] = [-P_0 - 4P_1 + 5P_\infty] \end{aligned}$$

Hence

$$[R_1 + R_2 + R_3 + P_0 + 4P_1 - 8P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + P_0 + 4P_1 - 8P_\infty, \text{ so } f \in \mathcal{L}(8P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have  $\text{ord}_{P_1} f = 4$  hence  $a = b = c = d = 0$  so  $f = ey$  we have  $e \neq 0$  otherwise  $\text{ord}_{P_1} f \neq 4$  which is absurd. At points  $R_i$ , we have  $0 = ey$  hence  $y = 0$  and therefore  $0 = x^4(x-1)^4$  either  $x = 0$  or  $(x-1)^4 = 0$  we find the points  $P_0$  and  $P_1$  which is absurd.

5<sup>th</sup> case  $m = 4$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 4j(P_0) = -4j(P_1) = [-4P_1 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 4P_1 - 7P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 4P_1 - 7P_\infty, \text{ so } f \in \mathcal{L}(7P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2}$$

We have  $\text{ord}_{P_1} f = 4$  hence  $a = b = c = d = 0$  and therefore one of  $f = 0$  which is absurd.

6<sup>th</sup> case  $m = 5$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 5j(P_0) = (11-6)j(P_0) = 11j(P_0) - 6j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 6j(P_0) = -6j(P_0) = [-6P_0 + 6P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 6P_0 - 9P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 6P_0 - 9P_\infty, \text{ so } f \in \mathcal{L}(9P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have  $\text{ord}_{P_0} f = 6$  hence

$a = b = c = d = e = 0$  so  $f = 0$  which contradicts the fact that  $\text{ord}_{P_0} f = 6$  absurd.

7<sup>th</sup> case  $m = 6$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 6j(P_0) = (11-5)j(P_0) = 11j(P_0) - 5j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 5j(P_0) = -5j(P_0) = [-5P_0 + 5P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 5P_0 - 8P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 5P_0 - 8P_\infty, \text{ so } f \in \mathcal{L}(8P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey$$

We have  $\text{ord}_{P_0} f = 5$ , hence  $a = b = c = d = e = 0$ ; so  $f = 0$  which contradicts the fact that  $\text{ord}_{P_0} f = 5$  absurd.

8<sup>th</sup> case  $m = 7$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 7j(P_0) = (11-4)(jP_0) = 11j(P_0) - 4j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 4j(P_0) = -4j(P_0) = [-4P_0 + 4P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 4P_0 - 7P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 4P_0 - 7P_\infty, \text{ so } f \in \mathcal{L}(7P_\infty) \text{ and as } \mathcal{L}(7P_\infty) = \mathcal{L}(6P_\infty)$$

So one of  $R_i$  should be equal to  $P_\infty$  which is absurd.

9<sup>th</sup> case  $m = 8$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 8j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 8j(P_0) = -8j(P_1) = [-8P_1 + 8P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 8P_1 - 11P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 8P_1 - 11P_\infty, \text{ so } f \in \mathcal{L}(11P_\infty)$$

According to the lemma 2, we have:

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} + d \frac{x(x-1)}{y^2} + ey + e_1 \frac{x^3(x-1)^3}{y^7} + e_2 x$$

We have  $\text{ord}_{P_1} f = 8$ , we must have  $a + e_2 = 0$  and  $b = c = d = e = e_1 = 0$ ; so  $f = e_2(x-1)$  and therefore one of  $R_i$  should be equal to  $P_1$  which is absurd.

10<sup>th</sup> case  $m = 9$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 9j(P_0) = (11-2)(jP_0) = 11j(P_0) - 2j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - 2j(P_0) = -2j(P_0) = -2[P_0 - P_\infty] = [-2P_0 + 2P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + 2P_0 - 5P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + 2P_0 - 5P_\infty, \text{ so } f \in \mathcal{L}(5P_\infty)$$

According to the lemma 2, we have  $f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$

We have  $\text{ord}_{P_0} f = 2$  hence  $a = b = 0$  so  $f = c \frac{x^2(x-1)^2}{y^5}$  we have  $c \neq 0$  otherwise  $\text{ord}_{P_0} f \neq 2$  ce qui est absurde. At points  $R_i$ , we have  $0 = c \frac{x^2(x-1)^2}{y^5}$  hence  $x^2(x-1)^2 = 0$  either  $x = 0$  or  $x = 1$  we find the points  $P_0$  and  $P_1$  which is absurd.

11<sup>th</sup> case  $m = 10$

The formula (t) becomes

$$[R_1 + R_2 + R_3 - 3P_\infty] = 10j(P_0) = (11-1)j(P_0) = 11j(P_0) - j(P_0)$$

According to the corollary 1 we have:

$$[R_1 + R_2 + R_3 - 3P_\infty] = 11j(P_0) - j(P_0) = -j(P_0) = [-P_0 + P_\infty]$$

Hence

$$[R_1 + R_2 + R_3 + P_0 - 4P_\infty] = 0$$

There exists a rational function  $f$  such that

$$\text{div}(f) = R_1 + R_2 + R_3 + P_0 - 4P_\infty, \text{ so } f \in \mathcal{L}(4P_\infty)$$

And therefore

$$f = a + b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$$

$$\text{ord}_{P_0} f = 1 \implies a = 0; \text{ so } f = b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5}$$

$b \neq 0$  otherwise  $\text{ord}_{P_0} f \neq 1$ ; which is absurd.

$c \neq 0$  sinon  $f \in \mathcal{L}(3P_\infty)$ ; which is absurd.

At points  $R_i$ , we have

$$0 = b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} :$$

$$b \frac{y^3}{x(x-1)} + c \frac{x^2(x-1)^2}{y^5} = 0$$

$$\iff y^8 = -\frac{c}{b} x^3 (x-1)^3$$

$$\iff y = \left( -\frac{c}{b} (x(x-1))^3 \right)^{\frac{1}{8}}$$

On the other hand, we have:

$$y^{11} = x^4 (x-1)^4$$

$$\iff \left( -\frac{c}{b} (x(x-1))^3 \right)^{\frac{11}{8}} = x^4 (x-1)^4$$

$$\iff \left( -\frac{c}{b} \right)^{\frac{11}{8}} (x(x-1))^{\frac{33}{8}} = x^4 (x-1)^4$$

$$\iff \left( -\frac{c}{b} \right)^{\frac{11}{8}} (x(x-1))^{\frac{32}{8}} (x(x-1))^{\frac{1}{8}} = x^4 (x-1)^4$$

$$\iff \left( -\frac{c}{b} \right)^{\frac{11}{8}} x^4 (x-1)^4 (x(x-1))^{\frac{1}{8}} = x^4 (x-1)^4$$

$$\iff \left( -\frac{c}{b} \right)^{\frac{11}{8}} (x(x-1))^{\frac{1}{8}} = 1$$

$$\iff \left( -\frac{c}{b} \right)^{11} x(x-1) = 1$$

$$\iff x(x-1) = \left(-\frac{b}{c}\right)^{11}$$

$$\iff x(x-1) = -\left(\frac{b}{c}\right)^{11}$$

We thus find a family of points:

$$S_1 = \left\{ \left( x, (\alpha(x(x-1))^3)^{\frac{1}{8}} \right) \mid \alpha \in \mathbb{Q}^* \text{ and } x \text{ is the root of the equation } x(x-1) = \frac{1}{\alpha^{11}} \right\}$$

## 4. Conclusion

Our note focuses on the determination of algebraic points on the curve  $\mathcal{C}_{4,4}(11)$  of affine equation  $y^{11} = x^4(x-1)^4$ . The curve  $\mathcal{C}_{4,4}(11)$  is a special case of the quotients of Fermat curves. In this note we have explicitly determined the algebraic points of degree at most 3 on the curve  $\mathcal{C}_{4,4}(11)$  on  $\mathbb{Q}$ . To do this we determined the quadratic points and the cubic points on  $\mathcal{C}_{4,4}(11)$  on  $\mathbb{Q}$ .

It seems possible to determine explicitly the algebraic points of any given degree on the curve  $\mathcal{C}_{4,4}(11)$ .

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