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Decompositions of the Covariance Matrix of the Discrete Brownian Bridge: New Fast Constructions of Discrete Brownian Motions and Brownian Bridges

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Abstract: Fast constructions from the Brownian motion and Brownian bridge are required in many applications such as Quasi-Monte Carlo simulations and statistical inferences on stochastic processes. The simple method for construction of discrete Brownian motion is a step-by-step method of computing the cumulative sum of i.i.d. normal variables. The construction of a N dimensional discrete Brownian motion (or a N-1 dimensional discrete Brownian bridge) that require at most O(Nlog$N$) floating point operations(flops) is called fast one. Discrete Brownian motion can be also constructed using decompositions of its covariance matrix and the method based on eigenvalue decomposition not only shows superior performances in many simulations to the step-by-step method but also becomes a fast construction. Usually the discrete Brownian bridge can be constructed from the discrete Brownian motion using the linear relationship between them. In this paper, the inverse of the covariance matrix for the discrete Brownian bridge is computed. The explicit expression of eigenvalue decomposition for the covariance matrix is given. Using it, a fast construction of the discrete Brownian Bridge is derived. The LDU (Lower-Diagonal-Upper) decompositions of the covariance matrices for the discrete Brownian motion and Brownian Bridge are obtained, respectively. The constructions of the discrete Brownian motion and Brownian bridge derived from these decompositions are fast ones and have step-by-step types. It is interesting that the discrete Brownian bridge is constructed as the cumulative sum of normal variables. Performances of the step-by-step method and methods using LDU and eigenvalue decompositions are compared through simulation results on the maximum distributions of the Brownian motion and Brownian bridge. Finally, an inserting method for construction of discrete Brownian motion using eigenvalue decompositions which requires O(Nlog(log$N$)) flops is proposed. The new fast constructions could be significant in Quasi-Monte Carlo simulations require high accuracy.

Keywords: Brownian Motion, Brownian Bridge, LDU Decomposition, Eigenvalue Decomposition, Quasi-Monte Carlo

1. Introduction

Let $B=\{B(t);\ t\in[0,\ 1]\}$ be a standard Brownian motion(simply a Brownian motion) and $B(0)=0$. B is an independent-increment process and $B(t)-B(s)$, $0\le s\le t\le 1$ follows the normal distribution with mean 0 and variance $t-s$, i.e., $B(t)-B(s)\sim N(0, t-s)$. The expectation and covariance of the Brownian motion $B=\{B(t);\ t\in[0,1]\}$ are respectively

$$EB(t)=0,\ Cov(B(s),\ B(t))=s,\ 0\le s\le t\le 1. \tag{1}$$

We set

$$BB(t) = B(t) - tB(t), \tag{2}$$

then $BB=\{BB(t);\ t\in[0,\ 1]\}$ is a standard Brownian bridge(simply a Brownian bridge). The Brownian bridge is a Brownian motion given $B(1)=0$, i.e., $BB \triangleq B|_{B(1)=0}$, where $\triangleq$ means equality in distribution. The expectation and covariance of the Brownian bridge are respectively

$$EBB(t)=0,\ Cov(BB(s),\ BB(t))= s(1-t),\ 0\leqq s\leqq t\leqq 1. \tag{3}$$

$(B(t_1),\ldots, B(t_N))$ and $(BB(t_1),\ldots, BB(t_{N-1}))$ will be called a discrete Brownian motion and a discrete Brownian bridges, respectively for $0=t_0<t_1<\ldots<t_{N-1}<t_N=1$. The case that the t_i are evenly spaced is the most important one from the practical point of view. Let $B_k=B(t/N)$, $k=1,\ldots, N$ and $BB_k=BB(t/N)$, $k=1,\ldots,N-1$. $B_N=(B_1,\ldots, B_N)^T$ is called the N dimensional discrete Brownian motion and $BB_{N-1}=(BB_1,\ldots, BB_{N-1})^T$ is called the N-1 dimensional discrete Brownian bridge.

The simple method for construction of the discrete Brownian motion B_N is to compute the cumulative sum of i.i.d. normal variables of mean zero and variance $1/N$ and it is called a step-by-step method and it uses $O(N)$ flops.

From (1) and (3) the covariance matrices of B_N and BB_N are

$$\Sigma_B = N^{-1}\begin{pmatrix} 1 & 1 & 1 & & 1 & 1 & 1 \\ 1 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 1 & 2 & 3 & & 3 & 3 & 3 \\ & \vdots & & \ddots & & \vdots & \\ 1 & 1 & 1 & & N-1 & N-2 & N-2 \\ 1 & 2 & 3 & \cdots & N-2 & N-1 & N-1 \\ 1 & 2 & 3 & & N-2 & N-1 & N \end{pmatrix},$$

$$\Sigma_{BB} = N^{-2}\begin{pmatrix} N-1 & N-2 & N-3 & & 3 & 2 & 1 \\ N-2 & 2(N-2) & 2(N-3) & \cdots & 2\cdot 3 & 2\cdot 2 & 2 \\ N-3 & 2(N-3) & 3(N-3) & & 3\cdot 3 & 3\cdot 2 & 3 \\ & \vdots & & \ddots & & \vdots & \\ 3 & 2\cdot 3 & 3\cdot 3 & & (N-3)3 & (N-3)2 & N-3 \\ 2 & 2\cdot 2 & 3\cdot 2 & \cdots & (N-3)2 & (N-2)2 & N-2 \\ 1 & 2 & 3 & & N-3 & N-2 & N-1 \end{pmatrix},$$

respectively.

Both of B_N and BB_{N-1} follow the multidimensional normal distribution. If the covariance matrix Σ of the normal random vector Z is decomposed into $\Sigma=CC^T$, samples of Z can be constructed by CW, where W is a standard normal vector [12].

The methods using the decomposition of covariance matrix $\Sigma_B=C_BC_B^T$ use $O(N^2)$ flops.

Åkesson, F and Lehoczky, J. P [1] has shown the following eigenvalue decomposition of Σ_B.

$$\Sigma_B = T_B\Lambda_B T_B^T, \tag{4.}$$

where $\Lambda_B = diag(\lambda_1, \ldots, \lambda_N)$, $\lambda_i = (4N sin^2((i-1/2)\pi/(2N)))^{-1}$, $T_B = (t_{ij})_{N\times N}$, and $t_{ij} = 2/\sqrt{2N+1} sin(i(2j-1)\pi/(2N+1))$, using

$$\Sigma_B = N\begin{pmatrix} 2 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 1 \end{pmatrix}_{N\times N} \tag{5}$$

Scheicher, K [14] has shown the construction using eigenvalue decomposition in (4) can be computed using the fast sine transform, thereby using $O(N\log N)$ flops(in the case of $N=2^K$ for a certain natural number). Leobacher, G [8] proposed useful orthogonal transforms using $O(N\log N)$ flops.

Wang, X and Sloan, I. H [15] and Imai, J and Tan, K. S [6] have pointed the importance of the proper choice of C_B in the decomposition of Σ_B for problems arising from quasi-Monte Carlo pricing of financial derivatives.

Scheicher, K [14] has shown any orthogonal transformation of the discrete Brownian motion by step-by-step method give another sampling method of it in theoretical view of point.

From (2), the samples of the discrete Brownian bridge $BB_{N-1}=(BB_1,\ldots,BB_{N-1})^T$ are usually constructed by the linear transformation of the discrete Brownian motion $B_N=(B_1,\ldots,B_N)^T$ as follows.

$$BB_k = B_k - (k/N)B_N,\ k=1,\ldots, N-1 \tag{6}$$

To the best of our knowledge, there is no result of sampling BB_{N-1} by using decomposition of Σ_{BB}.

The method to insert discrete Brownian bridges into a discrete Brownian motion (called Brownian Bridge construction) has been considered [2, 4, 5, 9-11]. The inserting method is useful for sampling high dimensional discrete Brownian motion [8]. Larcher, G et al. [7] has proposed a method for finding good weights for several classes of functions and applied it to certain algorithms using the Brownian Bridge construction. In that cases inserted discrete Brownian bridges have been sampled by (6) from the discrete Brownian motion.

The inserting method using a decomposition of Σ_{BB} gives another decomposition of covariance matrix for high dimensional discrete Brownian motion and it is significant in Quasi-Monte Carlo simulations for financial derivatives. For a financial derivative, the efficiency of Quasi-Monte Carlo simulations depends crucially on the decomposition of covariance matrix of the discrete Brownian motion [15].

In this paper, the expressions of eigenvalue and LDU decompositions of the covariance matrices of the discrete Brownian motion and Brownian bridge are given and new fast constructions of the discrete Brownian motion and Brownian bridge using these decompositions are presented. It is suggested an inserting method for construction of discrete Brownian motion which requires $O(N\log(\log N))$ flops. The method is based on eigenvalue decompositions of the covariance matrices and the suitable dimension partition of the discrete Brownian motion and Brownian bridges.

This paper is organized as follows. In section 2, the inverse matrix of Σ_{BB} is computed and the explicit expression of the eigenvalue decomposition of Σ_{BB} is given. A new fast construction of BB_{N-1} using the decomposition is derived. In section 3, the explicit expressions of the LDU decompositions of Σ_{BB} and Σ_B are given and new fast constructions of BB_{N-1} and B_N are proposed, respectively. The construction algorithms use $O(N)$ flops same as the step-by-step method. In section 4, performances of the step-by-step method and methods using LDU and eigenvalue decomposition are compared by the simulation results on the maximum distribution of the Brownian motion and Brownian bridge. Finally, the fast construction of the high dimensional discrete Brownian motion using $O(N\log(\log N))$ flops is shown.

2. Eigenvalue Decomposition and a Construction Using It

Using Gaussian elimination method, calculate the inverse matrix of Σ_{BB} such as

$$\Sigma_{BB}^{-1} = N\begin{pmatrix} 2 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix}_{(N-1)\times(N-1)} \tag{7}$$

This tridiagonal matrix is different from (5) only in the dimension and the element of the last row and column.

Theorem 2.1. The eigenvalue decomposition of $\Sigma_{BB} = (\bar{\sigma}_{ij})_{(N-1)\times(N-1)}$ is as follows.

$$\Sigma_{BB} = T_{BB}\Lambda_{BB}T_{BB}^T, \tag{8}$$

where $\Lambda_{BB} = diag(\bar{\lambda}_1, \dots, \bar{\lambda}_N)$, $\bar{\lambda}_i = (4N\sin^2(i\pi/N))^{-1}$, $T_{BB} = (\bar{t}_{ij})_{N\times N}$, and $\bar{t}_{ij} = \sqrt{2/N}\sin(2ij\pi/N)$.

Proof. The vectors $\bar{t}_i = (\bar{t}_{1i}, \dots, \bar{t}_{N-1,i})^T$, i=1,…, N-1 are orthonormal basis of N-1 dimensional space [8].

Now, in the similar way to Åkesson, F and Lehoczky, J. P [1], prove

$$\Sigma_{BB}^{-1}\bar{t}_i = \bar{\lambda}_i^{-1}\bar{t}_i,\ i=1,\dots,\ N-1. \tag{9}$$

For the first component of the left side in (9),

$$N(2\bar{t}_{1i} - \bar{t}_{2i}) = \sqrt{2N}\left(2\sin\left(\frac{2i\pi}{N}\right) - \sin\left(\frac{4i\pi}{N}\right)\right)$$

$$=\sqrt{2N}\left(2\sin\left(\frac{2i\pi}{N}\right) - 2\sin\left(\frac{2i\pi}{N}\right)\cos\left(\frac{2i\pi}{N}\right)\right)$$

$$=2\sqrt{2N}\sin\left(\frac{2i\pi}{N}\right)\left(1-\cos\left(\frac{2i\pi}{N}\right)\right)$$

$$=4N\sin^2\left(\frac{2i\pi}{N}\right)\cdot\sqrt{\frac{2}{N}}\sin\left(\frac{2i\pi}{N}\right) = \bar{\lambda}_i^{-1}\bar{t}_{1i}.$$

and for the j^{th}($2\le j\le N$-2) component,

$$N(-\bar{t}_{j-1,i} + 2\bar{t}_{ji} - \bar{t}_{j+1,i})$$

$$= \sqrt{2N}\left(2\sin\left(\frac{2ij\pi}{N}\right) - \sin\left(\frac{2i(j-1)\pi}{N}\right) - \sin\left(\frac{2i(j+1)\pi}{N}\right)\right)$$

$$= \sqrt{2N}\left(2\sin\left(\frac{2ij\pi}{N}\right) - 2\sin\left(\frac{2ij\pi}{N}\right)\cos\left(\frac{2i\pi}{N}\right)\right)$$

$$= 2\sqrt{2N}\sin\left(\frac{2ij\pi}{N}\right)\left(1-\cos\left(\frac{2i\pi}{N}\right)\right)$$

$$= 4N\sin^2\left(\frac{2i\pi}{N}\right)\cdot\sqrt{\frac{2}{N}}\sin\left(\frac{2ij\pi}{N}\right) = \bar{\lambda}_i^{-1}\bar{t}_{ji}.$$

Finally, for the j^{th}(j=N-1) component,

$$N(-\bar{t}_{N-2,i} + 2\bar{t}_{N-1,i})$$

$$= \sqrt{2N}\left(-\sin\left(\frac{4i(N-2)\pi}{N}\right) + 2\sin\left(\frac{2i(N-1)\pi}{N}\right)\right)$$

$$= \sqrt{2N}\left(2\sin\left(\frac{4i\pi}{N}\right) - 2\sin\left(\frac{2i\pi}{N}\right)\right)$$

$$= -\sqrt{2N}\left(2\sin\left(\frac{2i\pi}{N}\right) - 2\sin\left(\frac{2i\pi}{N}\right)\cos\left(\frac{2i\pi}{N}\right)\right)$$

$$= 2\sqrt{2N}\sin\left(\frac{2i(N-1)\pi}{N}\right)\left(1-\cos\left(\frac{2i\pi}{N}\right)\right)$$

$$= 4N\sin^2\left(\frac{2i\pi}{N}\right)\cdot\sqrt{\frac{2}{N}}\sin\left(\frac{2i(N-1)\pi}{N}\right)$$

$$= \bar{\lambda}_i^{-1}\bar{t}_{N-1,i},$$

where, the equality

$$\sin\left(\frac{2ij\pi}{N}\right) = \sin\left(\frac{2i\pi - 2i(N-j)}{N}\right) = -\sin\left(\frac{2i(N-j)\pi}{N}\right).$$

is used.

Now, the fast construction of the discrete Brownian bridge using (8) is considered. Let $W_{N-1}=(w_1,\dots,w_{N-1})^T$ be a $N-1$ dimensional standard normal vector and $V_{N-1} = (v_1, \dots, v_{N-1})^T = \Lambda_{BB}^{1/2}W_{N-1}$, then the i^{th} component of $BB_{N-1}=(BB_1,\dots,BB_{N-1})^T$ is as follows.

$$BB_i = \frac{1}{\sqrt{2N}}\sum_{k=1}^{N-1}\sin\left(\frac{2ki\pi}{N}\right)v_k,\ \text{i=1,…, N-1.}$$

By a mapping F: $R^{N-1}\to R^{2N-2}$ defined as

$$Fs := (0, s_1, 0, s_2, \dots, 0, s_{N-1})^T \epsilon R^{2N-2},$$

$$s = (s_1, s_2, \dots, s_{N-1})^T \epsilon R^{N-1},$$

and setting

$$Y = (y_1, y_2, \dots, y_{2N-2})^T \coloneqq FV_{N-1} = (0, v_1, 0, v_2, \dots, 0, v_{N-1}),$$

then

$$\frac{BB_i}{\sqrt{2}} = \sqrt{\frac{2}{N}}\sum_{k=1}^{N-1}\sin\left(\frac{2ki\pi}{N}\right)v_k$$

$$= \sqrt{\frac{2}{N}}\sum_{k=1}^{N-1}sin\left(\frac{k(2i)\pi}{N}\right)y_k,\ i=1,\dots,N-1.$$

This is the $2i^{th}$ element of the discrete sine transform $T_{DS}Y$ in dimension $2N-2$ on the vector $Y=(y_1,\dots,y_{2N-2})^T$. Therefore

$$BB_{N-1} = \sqrt{2}HT_{DS}F\Lambda_{BB}^{1/2}W_{N-1}, \tag{10}$$

where H:$R^{2N-2}\to R^{N-1}$ is a mapping defined as

$$Ht := (t_2, t_4, \dots, t_{2N-2})^T \epsilon R^{N-1},$$

$$t = (t_1, \dots, t_{2N-2})^T \epsilon R^{2N-1}.$$

Multiplications by H, F and $\Lambda_{BB}^{1/2}$ can be done in $O(N-1)$ flops and multiplication by T_{DS} can be done in $O((N-1)\log(N-1))$ flops when $N=2^K$ for a certain natural number K [8]. So, the construction of BB_{N-1} by (10) uses $O(N\log N)$ flops.

3. LDU Decompositions and Constructions Using Them

The following theorem shows the expression of the LDU decomposition of Σ_{BB}.

Theorem 3.1. The LDU decomposition of $\Sigma_{BB} = (\bar{\sigma}_{ij})_{(N-1)\times(N-1)}$ is as follows.

$$\Sigma_{BB} = L_{BB} D_{BB} U_{BB}^T \tag{11}$$

where $D_{BB} = \operatorname{diag} d_{BB}$, $d_{BB}^T = (\bar{d}_1, \dots, \bar{d}_{N-1}) = ((2N)^{-1}, 2(3N)^{-1}, \dots, (N-1)N^{-2})$ and

$$L_{BB} = U_{BB} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{3} & & (-1)^{N-3}\frac{1}{N-1} & (-1)^{N-2}\frac{1}{N-1} \\ 0 & 1 & -\frac{2}{3} & \cdots & (-1)^{N-4}\frac{2}{N-2} & (-1)^{N-3}\frac{2}{N-1} \\ 0 & 0 & 1 & & (-1)^{N-5}\frac{3}{N-2} & (-1)^{N-4}\frac{3}{N-1} \\ & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{N-2}{N-1} \\ 0 & 0 & 0 & & 0 & 1 \end{pmatrix}$$

Proof. From (7)

$$N^{-1}\Sigma_{BB}^{-1} = \begin{pmatrix} 2 & -1 & 0 & & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & & 0 & 0 & 0 \\ & \vdots & & \ddots & & \vdots & \\ 0 & 0 & 0 & & 2 & -1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 2 & -1 \\ 0 & 0 & 0 & & 0 & -1 & 2 \end{pmatrix}_{(N-1)\times(N-1)}.$$

The LDU decomposition [13] of above tridiagonal matrix is as follows.

$$N^{-1}\Sigma_{BB}^{-1} = \tilde{L}\tilde{D}\tilde{L}^T,$$

where $\tilde{D} = \operatorname{diag}(\tilde{d}_1, \dots, \tilde{d}_{N-1})$,

$$\tilde{L} = \tilde{U} = \begin{pmatrix} 1 & 0 & 0 & & 0 & 0 \\ \tilde{t}_2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & \tilde{t}_3 & 1 & & 0 & 0 \\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & & \tilde{t}_{N-1} & 1 \end{pmatrix}_{(N-1)\times(N-1)},$$

$$\tilde{t}_i = -\frac{1}{\tilde{d}_i}, i = 2, \dots, N-1,$$

and

$$\begin{cases} \tilde{d}_1 = 2, \\ \tilde{d}_i = 2 - \frac{1}{\tilde{d}_{i-1}}, i = 2, \dots, N-1 \end{cases}.$$

By recursive calculation,

$$\tilde{d}_i = \frac{i+1}{i}, i = 1, \dots, N-1.$$

On the other hand, using $\Sigma_{BB} = (\tilde{L}^{-1})^T (N^{-1}\tilde{D}^{-1})\tilde{L}^{-1}$ give

$$D_{BB} = N^{-1}\tilde{D}^{-1} = \operatorname{diag}((2N)^{-1}, 2(3N)^{-1}, \dots, (N-1)N^{-2}$$

and

$$L_{BB}=\left(\tilde{L}^{-1}\right)^T=\begin{pmatrix}1 & \tilde{t}_1 & \tilde{t}_1\tilde{t}_2 & \cdots & \prod_{k=1}^{N-2}\tilde{t}_k & \prod_{k=1}^{N-1}\tilde{t}_k\\ 0 & 1 & \tilde{t}_2 & \cdots & \prod_{k=2}^{N-2}\tilde{t}_k & \prod_{k=2}^{N-1}\tilde{t}_k\\ 0 & 0 & 1 & & \prod_{k=3}^{N-2}\tilde{t}_k & \prod_{k=3}^{N-1}\tilde{t}_k\\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & \tilde{t}_{N-1}\\ 0 & 0 & 0 & & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & -\frac{1}{2} & \frac{1}{3} & & (-1)^{N-3}\frac{1}{N-1} & (-1)^{N-2}\frac{1}{N-1}\\ 0 & 1 & -\frac{2}{3} & \cdots & (-1)^{N-4}\frac{2}{N-2} & (-1)^{N-3}\frac{2}{N-1}\\ 0 & 0 & 1 & & (-1)^{N-5}\frac{3}{N-2} & (-1)^{N-4}\frac{3}{N-1}\\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -\frac{N-2}{N-1}\\ 0 & 0 & 0 & & 0 & 1\end{pmatrix}_{(N-1)\times(N-1)}.$$

Using (11), $BB_{N-1}=(BB_1,\ldots,BB_{N-1})^T$ can be constructed as follows.

$$BB_{N-1}=L_{BB}D_{BB}^{1/2}W_{N-1}.$$

Now, let $V_{N-1}=(v_1,\ldots,v_{N-1})^T=D_{BB}^{1/2}W_{N-1}$, where W_{N-1} is a $N-1$ dimensional standard normal random vector then $BB_{N-1}=(BB_1,\ldots,BB_{N-1})^T=L_{BB}V_{N-1}$ can be constructed as follows.

$$BB_{N-1}=v_{N-1},$$

$$BB_i=v_i-\frac{i}{i+1}BB_{i+1},\ i=N\text{-}2,\ N\text{-}3,\ldots,\ 1. \tag{12}$$

The following theorem shows the expression for the LDU decomposition of Σ_B.

Theorem 3.2. The LDU decomposition of $\Sigma_B=\left(\sigma_{ij}\right)_{N\times N}$ is as follows.

$$\Sigma_B=L_BD_BU_B^T, \tag{13}$$

where

$D_B=\operatorname{diag}d_B,\ d_B^T=(d_1,\ldots,d_{N-1})=((2N)^{-1},2(3N)^{-1},\ldots,(N-1)N^{-2},1)$ and

$$L_B=U_B^T=\begin{pmatrix}1 & -\frac{1}{2} & \frac{1}{3} & & (-1)^{N-2}\frac{1}{N-1} & (-1)^{N-1}\frac{1}{N}\\ 0 & 1 & -\frac{2}{3} & \cdots & (-1)^{N-3}\frac{2}{N-1} & (-1)^{N-2}\frac{2}{N}\\ 0 & 0 & 1 & & (-1)^{N-4}\frac{3}{N-1} & (-1)^{N-3}\frac{3}{N}\\ & \vdots & & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & -\frac{N-2}{N-1}\\ 0 & 0 & 0 & & 0 & 1\end{pmatrix}_{N\times N}.$$

This proof, being very similar to above theorem, is omitted.

Using (13), $B_N=(B_1,\ldots,B_N)^T$ can be constructed as follows

$$B_N=L_BD_B^{1/2}W_N.$$

Now, let $U_N=(u_1,\ldots,u_N)^T=D_{BB}^{1/2}W_N$, where W_N is a N dimensional standard normal random vector, then $B_N=(B_1,\ldots,B_N)^T=L_BU_N$ can be constructed as follows.

$$B_N=u_N,$$

$$B_i=u_i-\frac{i}{i+1}B_{i+1},\ i=N\text{-}1,\ N\text{-}2,\ldots,\ 1. \tag{14}$$

The construction methods of the discrete Brownian motion and Brownian bridge using (14) and (12) are called LDU decomposition methods, simply. It is easy to find that LDU decomposition methods are similar to the step-by-step method in structure and use $O(N)$ flops.

4. Performance Comparison and the New Fast Construction

First, performances of above construction methods for approximation of the maximum distribution are compared. The cumulative distribution functions of $U_B=\max_{t\in[0,1]}B(t)$, $M_B=\max_{t\in[0,1]}|B(t)|$, $U_{BB}=\max_{t\in[0,1]}BB(t)$ and $M_{BB}=\max_{t\in[0,1]}|BB(t)|$ are represented as follows, respectively [3].

$$G_B^+(x)=P(U_B\le x)=2\Phi(x)-1,$$

$$\mathrm{G}_B(x)=P(M_B\le x)=\sum_{k=-\infty}^{\infty}(-1)^k\left[\Phi\left((2k+1)x\right)-\Phi\left((2k-1)x\right)\right],x>0,$$

$$G_{BB}^+(x)=P(U_{BB}\le x)=1-\exp(-2x^2),x>0,$$

$$\mathrm{G}_{BB}(x)=P(M_{BB}\le x)=2\sum_{k=-\infty}^{\infty}(-1)^k\exp(-2k^2x^2),x>0,$$

where Φ is the standard normal distribution function, i.e., $\Phi(x) = (2\pi)^{-1/2}\int_{-\infty}^{x}\exp(-z^2/2)dz$.

Generating R samples $B_{Nr}=(B_{1r},\ldots,B_{Nr})^T$, $r=1,\ldots, R$ and $BB_{N-1,r}=(BB_{1r},\ldots,BB_{N-1,r})^T$, $r=1,\ldots, R$ using step-by-step method, LDU decomposition and eigenvalue decomposition methods, respectively, $U_{B_{Nr}} = \max_{1\le j\le N} B_{jr}$, $M_{B_{Nr}} = \max_{1\le j\le N}|B_{jr}|$, $U_{BB_{N-1,r}} = \max_{1\le j\le N-1} BB_{jr}$ and $M_{BB_{N-1,r}} = \max_{1\le j\le N-1}|BB_{jr}|$, $r=1,\ldots, R$ are calculated.

Let denote by $\tilde{G}_B^+(x)$, $\tilde{G}_B(x)$, $\tilde{G}_{BB}^+(x)$ and $\tilde{G}_{BB}(x)$, the empirical distribution functions of them, respectively. For three methods, norm square errors

$$\|\Delta\tilde{G}\|^2 = \int_0^T|\tilde{G}(x) - G(x)|^2dx$$

calculated using MATLAB are presented in Table 1.

Table 1. *The norm square errors of the empirical distribution functions in the case of N=1024, R=10^5, and T = 4.*

Methods	$\|\Delta\tilde{G}_B^+\|^2$	$\|\Delta\tilde{G}_B\|^2$	$\|\Delta\tilde{G}_{BB}^+\|^2$	$\|\Delta\tilde{G}_{BB}\|^2$
step-by-step	0.0176	0.0196	0.0185	0.0181
LDU decomposition	0.0168	0.0178	0.0194	0.0187
Eigenvalue decomposition	0.0147	0.0143	0.0172	0.0167

Table 1 shows that the eigenvalue decomposition method has the smallest error in all cases and the LDU decomposition method is superior to the step-by-step method in 3 cases except of approximation of G_{BB}^+.

Finally, a method to construct the $N=N_1N_2$ dimensional discrete Brownian motion by inserting N_1 discrete Brownian bridges of dimension N_2-1 into a N_1 dimensional discrete Brownian motion is considered. Let denote by

$$B_{N_1}^{(0)} = \left(B_1^{(0)},\ldots,B_{N_1}^{(0)}\right)^T,$$

$$BB_{N_1-1}^{(j)} = \left(BB_1^{(j)},\ldots,B_{N_2-1}^{(j)}\right)^T, \text{j=1,…, N}_1,$$

N_1 dimensional discrete Brownian motion and N_1 discrete Brownian bridges of dimension N_2-1, respectively. Then, the $N=N_1N_2$ dimensional discrete Brownian motion $B_N=(B_1,\ldots,B_N)^T$ can be constructed as

$$B_{jN_2} = B_j^{(0)},\ j=1,\ldots,\ N_1, \tag{15}$$

and for $l = (j-1)N_2 + k, j=1,\ldots,N_1, k=1,\ldots, N_2-1$,

$$B_l = \frac{N_2-k}{N_2}B_{(j-1)N_2} + \frac{k}{N_2}B_{jN_2} + \frac{1}{\sqrt{N_1}}BB_k^{(j)}, \tag{16}$$

where $B_0=0$.

Constructing $B_{N_1}^{(0)}$ and $BB_{N_1-1}^{(j)}$ by using the eigenvalue decomposition allows the above construction using (15) and (16) of B_N use $O(N_1\log N_1+N_1N_2\log N_2)$ flops. If we set $N_2=\log N$, it turns to $O(N\log(\log N))$.

5. Conclusion

In this paper, explicit expressions of the eigenvalue and LDU decompositions of the covariance matrix of the discrete Brownian bridge have been given. And explicit expression of LDU decomposition of the covariance matrix of the discrete Brownian motion has been also given. New fast construction algorithms for sampling the discrete Brownian motion and Brownian bridge using these decompositions have been proposed.

The new fast constructions using LDU decomposition can be useful in certain simulations. Wang, X and Sloan, I. H [15] noted that if a decomposition works well for a given financial derivative using a QMC methods, then for every other decomposition there is another financial derivative which can be priced with exactly the same result. Also Table 1 shows that the LDU decomposition method is superior to the step-by-step method in most cases of approximation of the maximum distribution.

Further, the construction of the high dimensional discrete Brownian motion using $O(N \log(\log N))$ flops by inserting the discrete Brownian bridges constructed using the fast construction into the discrete Brownian motion have been presented. Inserting the discrete Brownian bridges constructed using different methods could provide different constructions of the discrete Brownian motion. For example, the discrete Brownian motion can be constructed alternatively by inserting discrete Brownian bridges constructed using LDU decomposition. It can be seen from Wang, X and Sloan, I. H [15] that every construction has a certain payoff function for which it is especially well suited. Several construction methods have been proposed and finding a suitable payoff function for each construction can be our future research. The new fast constructions will be useful in many Quasi-Monte Carlo simulations that require high accuracy.

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