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Locally H-closed Spaces, Subspaces and Their Extensions

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Abstract: The primary goal is to characterize Locally H-closed spaces (LHC), by conditions on the remainders of their extensions. These spaces are also characterized using subspaces and their extensions as well. Characterizing these classes of spaces using the remainders of the subspaces in their extensions provide characterizations of them in terms of their boundaries. Recently, the authors have proved that these results give necessary and sufficient conditions for the space to be compact A number of equivalences are proved for Hausdorff (Urysohn) [regular] spaces. These results lead to similar characterizations of Locally Urysohn-closed (LUC) as well as Locally regular-closed (LRC) spaces. Some of these equivalent properties generalize a number of existing results on these topics. In the present article it is shown that if X is a Hausdorff LHC space then each closed set is an intersection of regularly open sets as well as each closed set is an intersection of semi-closed neighborhoods. In 1969 Porter and Thomas had shown that in a Hausdorff space a locally H-closed subspace is the intersection of an open set and a closed set. In this article, it is shown that a space X is LHC if and only if every nonempty proper regularly closed subset of X is LHC.

Keywords: H-closed Extensions, Locally H-closed, θ-closure, u-closure, s-closure, θ-rigid

1. Introduction

A Hausdorff topological space is defined to be *H-closed* if it is a closed subset of any Hausdorff space in which it is embedded, and is *locally H-closed (LHC)* if each of its points has an H-closed neighborhood. In 1991, several properties of LHC spaces are established [7]. It includes the following:

Theorem 1.1 A Hausdorff space X is LHC if and only if the remainder $\kappa X \setminus X$ is a θ-closed subset of every (some) H-closed space in which it is embedded, where κX is the Katětov extension of X.

In this section background and relevance of the study of LHC spaces in relation to the previous results of such spaces are given. It is not exhaustive, but those results relevant to the study here are selected.

A subset B of a Hausdorff space is quasi Hausdorff closed (QHC) (quasi Urysohn closed ((QUC))) [quasi regular closed (QRC) if each filter base Ω on B satisfies $B \cap adh_\theta\Omega(B \cap adh_u\Omega)[B \cap adh_s\Omega] \neq \emptyset$. (Definitions of the different adherence of a filter base indicated here are given later in this section.) It is shown that the following four statements are aquivalent for a Hausdorff (Urysohn) [regular] space.

1. X is LHC (LUC)[LRC];
2. $\kappa X \setminus X$ is QHC (QUC) [QRC] in κX;
3. $cl_\theta(\kappa X \setminus X)(cl_u(\kappa X \setminus X)cl_s(\kappa X \setminus X)) = \bigcup_{\kappa X \setminus X} cl_\theta\{x\}(cl_u\{x\})[cl_s\{x\}]$;
4. $\kappa X \setminus X$ is θ-closed (u-closed) [s-closed],

Where $cl_\theta A(cl_u A)[cl_s A]$ is the θ-closure (u-closure) [s-closure] of a subset A of a topological space. Some of the results in this article generize some earlier results by Espelie, Joseph and Kwack [3, 4, 11, 12]. In an H-closed space, $adh_\theta\Omega$ is shown to be QHC relative to X for any filterbase Ω. In an Urysohn closed space, $adh_u\Omega$ is shown to be QUC (quasi Urysohn closed) relative to X for any filterbase Ω. This last result is a solution to a generalization of a problem left open in 1981 [4]. Other genralizations include the following: If Ω, Γ are filterbases and $adh_\theta\Omega \cap adh_\theta\Gamma = \emptyset$, there are $V \in \bigcup_\Omega \sum F, W \in \bigcup_\Gamma \sum F$, such that $V \cap W = \emptyset$. If $adh_\theta\Omega \cap adh_u\Gamma = \emptyset$, there are $V \in \bigcup_\Omega \sum F, W \in \bigcup_\Gamma \Lambda F$, with $V \cap W = \emptyset$. If $adh_u\Omega \cap adh_u\Gamma = \emptyset$, there are $V \in \bigcup_\Omega \Lambda F, W \in \bigcup_\Gamma \Lambda F$, such that $V \cap W = \emptyset$. If X is $H(i)(U(i))$ then $adh_\theta\Omega(adh_u\Omega)$ is QHC (QUC) relative to X, (where $\sum F$ represents the open sets about F and

ΛF represents the collection of open sets containing a closed neighborhood of F). The concepts of *locally regularly-closed (LRC)* and *locally H-set (LHS)* are introduced and shown to be equivalent to LHC, in a Hausdorff space.

Concepts mentined above, including the concepts of θ-closure, u-closure, s-closure of a set and a θ-rigid subset etc. all are defined later in this section.

The following result was established by Espelie and Joseph [3].

Theorem 1.2 [3] If $A \subset X$ is θ-rigid, then $cl_\theta A = \bigcup_A cl_\theta\{x\}$.

The proof of the next theorem is easy and is omitted.

Theorem 1.3 For x, y in a space $X, x \in cl_\theta\{y\}$ if $y \in cl_\theta\{x\}(x \in cl_u\{y\}$ if $y \in cl_u\{x\})[x \in cl_s\{y\}$ if $y \in cl_s\{x\}]$.

An improvement of this result as well as additional results for (QUC) [QRC] subsets is the following.

Theorem 1.4 If $A \subset X$ is QHC (QUC) [QRC], then $cl_\theta A = \bigcup_A cl_\theta\{x\}(cl_u A = \bigcup_A cl_u\{x\})[cl_s A = \bigcup_A cl_s\{x\}]$.

Proof Let A be QHC (QUC) [QRC], $x \in cl_\theta A(x \in cl_u A)[x \in cl_s A]$. Suppose there is no $y \in A$ such that $x \in cl_\theta\{y\}(x \in cl_u\{y\})[x \in cl_s\{y\}]$. Then A is not QHC (QUC) [QRC]. Hence $cl_\theta A = \bigcup_A cl_\theta\{x\}(cl_u A = \bigcup_A cl_u\{x\})[cl_s A = \bigcup_A cl_s\{x\}]$.

The following Theorem was established in 1981 [12].

Theorem 1.5 The following statements are equivalent for a Hausdorff space X.

1. The space X is LHC;
2. $\kappa X \setminus X$ is a θ-closed subset of κX;
3. $\kappa X \setminus X$ is a θ-rigid subset of κX.

In this paper, the above Theorem is extended to the following. Note that if X is Urysohn (regular), then $(\kappa X \setminus X)$ is Urysohn (regular), being a descrete set.

Theorem 1.6 The following statements are equivalent for a Hausdorff (Urysohn) [regular]space X.

1. The space X is LHC;
2. $(\kappa X \setminus X)(\mu X \setminus X)$ is a θ-closed (u-closed) [s-closed] subset of κX (μX), for every H-closed extension μX of X;
3. $\kappa X \setminus X(\mu X \setminus X)$ is a QHC (QUC) [QRC] subset of $\kappa X(\mu X)$, for every H-closed extension μX of X;
4. The equation $cl_\theta(\kappa X \setminus X) = \bigcup_{\kappa X \setminus X} cl_\theta\{x\}(cl_u(\kappa X \setminus X) = \bigcup_{\kappa X \setminus X} cl_u\{x\})[cl_s(\kappa X \setminus X) = \bigcup_{\kappa X \setminus X} cl_s\{x\})]$ holds.

Following are the definitions of some of the concepts which are used in this article. Most of the definitions which are stated here, involving extensions of spaces, can be found in [21]. A space Y is called an *extension* of X if X is a dense subspace of Y. A subset A of a space X is *regularly closed* if it is the closure of an open set or equivalently, $A = cl(intA)$, where $intA$ is the interior of A and clA is the closure of A.The concept of θ-closure of a set was introduced by Veličko [22]. In the study of H-closed spaces, this concept helps to replace open filter base with arbitrary filter base as can be seen easily. If $A \subseteq X$, the *θ-closure of* A, denoted as $cl_\theta A = \{x : A \cap clV \neq \emptyset, \forall V \in \Sigma\{x\}\}$, where $\Sigma\{x\}$ represents the set of all open sets containing x. A set A is *θ-closed* if $A = cl_\theta A$. It should be noted that θ-closure of a set need not be θ-closed. If Ω is a filter base, *θ- adherence of* Ω, $adh_\theta \Omega = \bigcap_{F \in \Omega} cl_\theta F$. A set $A \subseteq X$ is said to be *θ-rigid*, if each filter base Ω on X satisfying the property that $F \cap clV \neq \emptyset$, for all $F \in \Omega$ and $V \in \Sigma(A)$ also satisfies that $adh_\theta \Omega \cap A \neq \emptyset$ [2]. It is shown that $cl_\theta(A) = \bigcup_A cl_\theta\{x\}$ for any θ-rigid $A \subset X$ and thus a θ-rigid set A is θ-closed in any Hausdoff space since such spaces satisfy $cl_\theta\{x\} = \{x\}$ [3]

The concept of u-closure of set was used by Joseph to study, among others, compact spaces as well as Urysohn-closed spaces [10]. Let $A \subseteq X$. The *u-closure* of A, denoted as $cl_u(A)$, = $\{x : clV \cap A \neq \emptyset, V \in \Lambda(x)\}$, where $\Lambda(x)$ represents the collection of open sets containing a closed neighborhood of x. A space is Urysohn if $cl_u(x) = \{x\}$. Also it is shown that $x \in cl_u(A)$ if and only if $cl_u(V) \cap A \neq \emptyset$ for every $V \in \sum(x)$ [4]. The concepts of u-adherence of a filter base Ω and a u-rigid subset are defined similar to the concepts of θ- adherence of a filter base and of a θ-rigid subset. The *u-adherence* of a filter base Ω, $adh_u \Omega = \bigcap_{F \in \Omega} cl_u(F)$. A set $A \subseteq X$ is said to be *u-rigid* if each filter base Ω on X satisfying the property that $F \cap clV \neq \emptyset$ for all $F \in \Omega$ and $V \in \Lambda(A)$, satisfies that $A \cap adh_u \Omega \neq \emptyset$ [4].

Herrington called a family of open sets, $\mathcal{G}$, a shrinkable family of open sets about a point $x \in X$ if for each $U \in \mathcal{G}$, there is a $V \in \mathcal{G}$ such that $x \in U \subseteq clU \subseteq V$ [8]. A point x is in the *s-closure* of $A \subset X$, denoted as $x \in cl_s(A)$, if and only if $V \cap A(cl_s(V) \cap A) \neq \emptyset$ for every $V \in \mathcal{G}(x)(V \in \sum(x))$, where $\mathcal{G}(x)$ is a shrinkable family of open sets around x. Herrington defined a point $x \in X$ to be in the s-adherence of a filterbase $\mathcal{F}$, denoted as $x \in adh_s \mathcal{F}$, if for each shrinkable family $\mathcal{G}$ of open sets about x and $F \in \mathcal{F}$, there is a $V \in \mathcal{G}$ such that $F \cap V \neq \emptyset$ [8]. That is, $adh_s \Omega = \bigcap_\Omega cl_s(F)$[8]. It was then proved that a regular space is regular-closed if and only if each filterbase on the space has non-empty s-adherence.

Dickman and Porter showed that a θ-rigid subset of any space is Quasi-H-closed relative to the space and that a θ-rigid subset of a Hausdorff space is θ-closed, where a set $A \subseteq X$ is *Quasi H-closed (QHC) relative to* X if each filter base Ω on X satisfies $A \cap adh_\theta \Omega \neq \emptyset$ [1]. If X, not necessarily Hausdorff, is QHC relative to X, X is an *$H(i)$ space* [19]. A Hausdorff $H(i)$ space is an H-closed space and a QHC relative to X subset is called an *H-set*, if X is Hausdorff [23]. Joseph showed that if X is an $H(i)$ space and $A \subseteq X$, $cl_\theta A$ is QHC relative to X [11]. Note that a QHC relative to X set need not be H-closed as a subspace. Porter and Tikoo pointed out that θ-closed subset of an H-closed space need not be H-closed [20]. However, Joseph proved that θ-closed subset of an $H(i)$-space is θ-rigid [12]. Consequently, the following Theorem was immediate.

Theorem 1.7 In an H-closed space, a subset is θ-closed if and only if it is θ-rigid.

Espelie et al established several comparative properties of $cl_\theta A$ and $cl_u A$, for a subset A of a QHC space X [4]. It was shown that (1) if $X$is QHC and if $A \subset X(\Omega$ a filter base on $X)$, then $cl_u(A)(ad_u \Omega)$ is QHC relative to X; (2) If A is QHC relative to a space X, then $cl_\theta A \subset \cup_A cl_u(x)$; (3) Every θ-rigid subset of a space is u-rigid, but a u-rigid subset need not be θ-rigid; (4) If A is a u-rigid subset of a space, then

$cl_u(A) = \cup_A cl_u(x)$ and hence (5) if A is a θ-rigid subset of a space, then $cl_u(A) = \cup_A cl_u(x)$.

As is clear from the discussion above, θ -closure of a set and θ-closed subsets of a space play significant roles in the study of H-closed spaces as well as in the study of LHC spaces. In view of the above stated Theorems, it is also clear that u-closure of a set and u-rigid subsets of a space could be effectively used to obtain new properties of LHC spaces.

As is stated in Theorem 1.7, in an H-closed space, a set is θ-closed if and only if it is θ-rigid. This observation paved the way for several new characterizations of H-closed spaces and LHC-spaces in terms of H-closed extensions. Theorem 1.5 is a consequence of the above Theorem and appeared in the same paper [12]. The main tools used in this study of LHC-spaces are H-closed extensions and the remainders of a space in H-closed extensions. We also use H-closed extensions of a subspace and the remainders to provide new characterizations of LHC spaces. Throughout this article, a space is considered to be Hausdorff. Also, in the sequel, μX represents any H-closed extension of X. Proofs of all results stated in sections 1 and 2 are given in subsequent sections as sections 1 and 2 are introductory sections.

2. H-closed Extensions

Let X be a Hausdorff space and let $X^*= X \cup \{\mathcal{U} : \mathcal{U}$ is a free open ultrafilter on $X\}$, where an open ultrafilter is called free if it does not converge. For each open set $V \subseteq X$, let $O(V) = V \cup \{\mathcal{U} \in X^* \smallsetminus X : V \in \mathcal{U}\}$. Then,

(i) $\{O(V) : V$ open in $X\}$ is an open base for a topology on X^*. X^* with this topology is an H-closed extension of X. This extension is called the Fomin extension σX [5].

(ii) X^* with the topology generated by the open base $\{V : V$ is open in $X\}$ $\cup\{V \cup \{\mathcal{U}\} : V \in \mathcal{U}, \mathcal{U} \in X^* \smallsetminus X\}$ is an H-closed extension of X, called the Katětov extension and is denoted by κX [14].

(iii) Let $\theta X = \{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on X $\}$. For each open $V \subseteq X$, let $o(V) = \{\mathcal{U} \in \theta X : V \in \mathcal{U}\}$. The family $\{o(V) : V$ open in X $\}$ is a base for a topology on θX, which is extremally disconnected, compact and Hausdorff [9].

It is to be noted that while the topologies in (i) and (ii) give extensions of a space, the topology generated by the base in (iii) provides an extremally disconnected and compact space associated with a given space X.

Obreanu introduced the concept of Locally H-closed spaces [16]. He studied one point H-closed extensions of such spaces. He proved that LHC spaces have one point H-closed extensions and that the set of one point H-closed extensions admits projective maximum and projective minimum. Also, he proved that the product of a non-void family of Hausdorff spaces is LHC if and only if all but finitely many members of the factor spaces are H-closed and each factor space is LHC. In 1991, Mike Girou studied LHC spaces, by investigating the remainders of H-closed extensions [7]. This was in line with the investigation done by Tikoo [22]. Tikoo studied remainders of H-closed extensions. He, noting that locally compact spaces always have compactifications with compact remainders, asked if LHC spaces were characterized by having H-closed extensions with H-closed remainders. Girou stated that the answer to this question was negative and proved that a space X is LHC if and only if any H-closed extension of X has a θ-closed remainder. However, note that this Theorem was an immediate consequence of Theorem 1.7 above as is stated in Theorems 1.5 and 1.6. Girou, also while investigating those H-closed spaces in which every H-closed sets are θ-closed, showed that such spaces are H-closed spaces which are Urysohn [7].

It is to be noted that many of the Theorems of Girou come as consequences of the observation by Joseph [12]. This fact is evident from Theorem 1.7. Also many new Theorems follow from the observation that the family of θ-closed subsets of an H-closed space coincides with the family of θ-rigid subsets.

In this article, several properties of H-closed spaces are extended to LHC-spaces. Also among other Theorems the following characterization of a LHC-space is established in terms of its subspaces:

Theorem 2.1 A space X is LHC if and only if every non-empty proper regularly closed subset of X is LHC.

Utilizing the function $\pi : \theta X \to \sigma X$ such that $\pi(\mathcal{U}) = \mathcal{U}$ for each free open ultrafilter $\mathcal{U}$ on X and $\pi(\mathcal{U}) = x$, where x is the unique convergent point of the fixed ultrafilter $\mathcal{U}$, the following characterization for a non-dense open subset of a LHC space to be LHC is given:

Theorem 2.2 Let X be LHC and $B \subset X$ be open and not dense in X. Then B is LHC if and only if $\pi^{-1}(X \smallsetminus B)$ is compact.

We also characterize LHC space X in terms of H-closed extensions of a subset B of X in the following Theorem.

Theorem 2.3 Let $B \subseteq X$. Then the following statements are equivalent for a Hausdorff space X.

1. *The space X is LHC;*
2. *$\mu B \smallsetminus B$ is a θ-closed subset of μB for every H-closed extension μB of B;*
3. *$\mu B \smallsetminus B$ is a θ-rigid subset of μB for every H-closed extension μB of B;*
4. *$\kappa B \smallsetminus B$ is a θ-rigid subset of κB;*
5. *$\kappa B \smallsetminus B$ is a θ-closed subset of κB.*

Characterizing LHC spaces in terms of $u - closre$ and θ-closure operators on the remainders of a Hausdorff space in an H-closed extension, we give the following Theorems:

Theorem 2.4 The following are equivalent for a Hausdorff space X:

1. *X is LHC;*
2. *$cl_\theta(\kappa X \smallsetminus X)$(respectively, $cl_\theta(\mu X \smallsetminus X)$ $=$ $\bigcup_{\kappa X \smallsetminus X} cl_\theta(x)$(respsectively, $\bigcup_{\mu X \smallsetminus X} cl_\theta(x)$);*
3. *$\kappa X \smallsetminus X$(respectively, $\mu X \smallsetminus X$) is u-rigid;*
4. *$\kappa X \smallsetminus X$)(respectively, $\mu X \smallsetminus X$) is an H-set in κX(respectively, μX).*

Theorem 2.5 . The following are equivalent for a Urysohn space X:

1. *X is is LHC;*
2. *$cl_u(\kappa X \smallsetminus X)$(respectively, $cl_u(\mu X \smallsetminus X)$) $=$ $\bigcup_{\kappa X \smallsetminus X} cl_u(x)$(respectively, $\bigcup_{\mu X \smallsetminus X} cl_u(x)$),.*

3. LHC-Spaces Through Subspaces and H-Closed Extensions

As stated in the Introduction, Theorem 1.5 can be extended to any H-closed extension μX of X as stated in Theorem 1.6. For the sake of completeness a direct proof of Theorem 1.6 is given in this section and is stated here as Theorem 3.1.

Theorem 3.1 . The following statements are equivalent for a Hausdorff space X.

1. *The space X is LHC;*
2. *$\mu X \setminus X$ is a QHC subset of μX for every H-closed extension μX of X;*
3. *$\mu X \setminus X$ is a θ-closed subset of μX for every H-closed extension μX of X;*
4. *$\mu X \setminus X$ is a θ-rigid subset of μX for every H-closed extension μX of X;*
5. *$\kappa X \setminus X$ is a θ-rigid subset of κX;*
6. *$\kappa X \setminus X$ is a θ-closed subset of κX.*

Proof $(1) \Rightarrow (3)$. Choose $x \in X$ and $V \in \Sigma_X(x)$ such that $\mathrm{cl}_X(V)$ is an H-closed subset of X. However, $\mathrm{cl}_{\mu X}(V) \cap X = \mathrm{cl}_X(V)$ and hence it follows that $\mathrm{cl}_{\mu X}(V) \cap X$ is an H-closed subset of X and hence is closed in μX, $\mathrm{cl}_{\mu X}(V) \cap X$ is a μX-closed neighborhood of x, and $(\mu X \setminus X) \cap \mathrm{cl}_{\mu X}(V) \cap X = \emptyset$. Hence $(\mu X \setminus X)$ is θ-closed in μX.

$(3) \Rightarrow (2)$. Any filterbase Ω in $(\mu X \setminus X)$ gives a filterbase in μX and hance has non-empty adherence. Since $(\mu X \setminus X)$ is θ-closed in μX, Ω has non-empty adherence in $(\mu X \setminus X)$.

$(2) \Rightarrow (3)$. Follows from [11].

$(3) \Rightarrow (4)$. Obvious.

$(4) \Rightarrow (5); (5) \Rightarrow (6)$. These follow from [12].

$(6) \Rightarrow (1)$. Suppose $\kappa X \setminus X$ is a θ-closed subset of κX and let $x \in X$. Choose an open subset $V \in \Sigma(x)$ of $x \in X$ such that $\mathrm{cl}_{\kappa X}(V) \cap (\kappa X \setminus X) = \emptyset$. It follows that $cl_{\kappa X}(V) \subseteq X$ and hence $\mathrm{cl}_{\kappa X}(V)$ is H - closed in X since $\mathrm{cl}_{\kappa X}(V) = \mathrm{cl}_X(V)$.

It is not difficult to see that "every" can be replaced with "some" in statements (2) and (3) since in that case, the proof of $(3) \Rightarrow (1)$ will follow in the same line as in the proof of $(6) \Rightarrow (1)$ with κX replaced with some H-closed extension ηX.

The following theorem can be proved using similar line of argument, using the facts that X is Urysohn (regular) if and only if κX is Urysohn (regular); $x \in cl_u(A)$ if and only if $cl_u(V) \cap A \neq \emptyset$ for every $V \in \sum(x)$, $x \in cl_s(A)$, if and only if $cl_s(V) \cap A \neq \emptyset$ for every $V \in \sum(x)$ and u-closure as well as s-closure of a set is closed. A space X is LUC (LRC) if each point has a Urysohn-closed (regular-closed) neighborhood.

Theorem 3.2 . The following statements are equivalent for a Hausdorff (Urysohn) [regular]space X.

1. *The space X is LHC (LUC) [LRC];*
2. *$\kappa X \setminus X$ is a QHC (QUC) [QRC] subset of κX;*
3. *$\kappa X \setminus X$ is a θ-closed (u-closed) [s-closed] subset of κX.*

Proof Clear from the above stated facts and with similar arguments as in the proof of Theorem 3.1.

Theorem 3.3 . A Hausdorff space X is LHC if and only if $cl_\theta(\mu X \setminus X) = \bigcup_{\mu X \setminus X} cl_\theta(x)$.

Proof Since the spaces are Hausdorff $cl_\theta(\mu X \setminus X) = \bigcup_{\mu X \setminus X} cl_\theta(x) = \mu X \setminus X$ because $cl_\theta\{x\} = \{x\}$. Thus the space is LHC because $\mu X \setminus X$ is θ closed. On the other hand, if X is LHC, then $\mu X \setminus X$ is θ-rigid so

$$cl_\theta(\mu X \setminus X) \subset cl_u(\mu X \setminus X) = \bigcup_{\mu X \setminus X} cl_\theta\{x\} = \mu X \setminus X,$$

since the spaces are Hausdorff, for each H-closed extension μX.

Theorem 3.4 . If μX is a Urysohn [regular] space, X is LHC if and only if $cl_u(\mu X \setminus X) = \bigcup_{\mu X \setminus X} cl_u(x)[cl_s(\mu X \setminus X) = \bigcup_{\mu X \setminus X} cl_s(x)]$.

Proof The proof is similar to the proof of Theorem 3.3, since in a Urysohn (regular) space, $\{x\} = cl_u\{x\}(\{x\} = cl_s\{x\})$.

Theorem 3.5 . A space is LHC if and only if $\kappa X \setminus X$(respectively, $\mu X \setminus X$) is an H-set.

Proof. If X is LHC, $\kappa X \setminus X$ (respectively, $\mu X \setminus X$) is θ closed in κX (respectively, μX) and thus is an H-set in κX (respectively, μX).

Conversely, suppose that $\kappa X \setminus X$ (respectively, $\mu X \setminus X$) is an H-set in κX (respectively, μX). Let Ω be a filter base in κX(respectively, μX). Then $adh_\theta \Omega \cap (\kappa X \setminus X) \neq \emptyset$ (respectively, $adh_\theta \Omega \cap (\mu X \setminus X) \neq \emptyset$). Therefore each filter base Ω which satisfies the property that $F \cap cl(W) \neq \emptyset$ for all $F \in \Omega$ and $W \in \Sigma(\kappa X \setminus X)$ (respectively, $W \in \Sigma(\mu X \setminus X)$) also implies that $(\kappa X \setminus X) \cap adh_\theta \Omega \neq \emptyset$ (respectively, $\mu X \setminus X) \cap adh_\theta \Omega \neq \emptyset$. Hence $\kappa X \setminus X$ (respectively, $\mu X \setminus X$) is θ-rigid and hence is θ- closed in $\kappa X \setminus X$ (respectively, $\mu X \setminus X$). Therefore, X is LHC.

The proof of the above theorem shows that *if $A \subseteq X$ is an H-set, then A is θ -rigid* and hence the following theorem is immediate.

Theorem 3.6. A Hausdorff space is LHC if and only if κX(respectively, μX)has an H-set in κX (respectively, μX).

Proof. Clearly $\kappa X \setminus X$ (respectively, $\mu X \setminus X$) is a θ closed subset of κX(respectively, μX) and is an H set of κX(respectively, μX). Moreover, $\kappa X \setminus X$ (respectively, $\mu X \setminus X$) is an H-set and hence is θ- rigid in κX(respectively, μX). THus X is LHC.

Our next proposition uses Theorem 3.1 to give another proof for Proposition 3.2 of [7].

Corollary 3.1 ([7], Proposition 3.2). *Let X be an H-closed space and B an open subset. Then the following are equivalent:*

1. *B is LHC.*
2. *$X \setminus B$ is a θ-closed subset of X;*
3. *BdB is a θ-closed subset of $\mathrm{cl}_X B$, where bdB is the boundary of the set B.*

Proof These conclusions are consequences of the facts that an H-closed subset of B is closed in X and $\mathrm{cl}_X B$ is an H-closed extension of B.

The next corollary is a new Theorem which is another consequence of the Theorem 3.1.

Corollary 3.2. Let X be an H-closed space and B an open subset. Then the following are equivalent:

1. *B is LHC.*
2. *$X \setminus B$ is a θ-rigid subset of X.*
3. *BdB is a θ-rigid subset of $cl_X B$.*

The corollaries 3.1 and 3.2 above can be stated in terms extensions of subspaces, as is shown in the following Theorem.

Theorem 3.7. Let X be H-closed. The following are equivalent for each open $A \subset X$:

1. *The subset A is LHC.*
2. *The subset $cl_X(A) \setminus A = bd_X(A)$ is a θ-closed subset of $cl_X(A)$.*
3. *The subset $cl_X(A) \setminus A = bd_X(A)$ is a θ-rigid subset of $cl_X(A)$.*

Proof $(1) \Rightarrow (2)$. Since A is an open subset of X $cl_X(A)$ is a regular closed subset of X and hence is an $H - closed$ extension of A. Therefore, in view of Theorem 3.1 (3), $cl_X(A) \setminus A$ is a θ-closed subset of $cl_X(A)$.

$(2) \Rightarrow (3)$. Follows from Theorem 3.1 (4)

$(3) \Rightarrow (1)$ This follows from the equivalence of (3) and (1) in Theorem 3.1.

Theorem 3.8 below improves on Proposition 3.9 of [7]. It is stated here as Corollary 3.3.

Theorem 3.8. If X is LHC and V is open and not dense in X, then $cl_X(V)$ is H-closed.

Proof Clearly $\mathrm{cl}_{\kappa X}(V) \cap X = \mathrm{cl}_X(V)$. It follows that $\mathrm{cl}_X(V)$ is H-closed in X, since $cl_X V$ is closed in H-closed extension κX of X.

Corollary 3.3 comes immediately from the above.

Corollary 3.3 [7]. *A regularly-closed subset of a LHC space is LHC.*

The Theorem 3.9 below improves on Theorem 4.15 in [11]. That result is stated here as Corollary 3.4.

Theorem 3.9 . If X is LHC and $A \subset X$ satisfies $cl_\theta^{\kappa X}(A) \subset X$, then $cl_\theta^X(A)$is an H-set.

Proof It is straightforward to show that $cl_\theta^{\kappa X}(A) = \mathrm{cl}_\theta^X(A)$, as in the proof of Theorem 3.1. Then $\mathrm{cl}_\theta^{\kappa X}(A)$ is an H-set, Theorem 4.15 of [11].

It should be noted that in the Theorem 3.8 and in Theorem 3.9, κX can be replaced by any H-closed extension μX of X, since for any H-closed extension μX of X, and an open $V \subset X$, $cl_{\mu X}(V) \cap X = cl_X(V)$. Sine X is LHC, it follows that $cl_{\mu X}(V) \cap (\mu X - X) = \emptyset$ and consequently, $cl_{\mu X}(V) = cl_X(V)$. Hence $cl_X(V)$ is H-closed in X.

Corollary 3.4 [11]. *If X is a QHC space and $K \subset X$, then $cl_\theta(K)$ is QHC relative to X.*

Corollary 3.5 The $\theta-$closure of any subset of an H-closed space is an H-set.

Theorem 3.9 is extended to the following characterization of LHC spaces.

Theorem 3.10 A space X is LHC if and only if for every $A \subseteq X, cl_\theta^{\kappa X}(A)$ (respectively, $cl_u^{\kappa X}(A)$) is an H-set in κX(in μX, when the closures are taken in μX).

Proof Clearly, since κX, as well as (μX), is H-closed, $cl_\theta(A)$ (respectively, $cl_u(A)$) is an H-set for any $A \subset X$ ([4],[11]). On the other hand, for the converse, since the assumption is true for for any $A \subseteq X$, let $A = X$. If $cl_\theta^{\kappa X}(X)$ (as well as $cl_u^{\kappa X}(X)$) is an H-set in κX, then for any $x \in X$ there exists $V \in \sum(x)$ such that $cl_X(V) \cap (\kappa X \setminus X) = \emptyset$. So $\kappa X \setminus X$ is θ-closed in κX and hence X is LHC. Replacing κX with μX, we can prove the result in the case of extension μX.

The Corollary 3.3 can be extended to the following to give a characterization of LHC-spaces.

Theorem 3.11 . A space X is LHC if and only if every non-empty proper regularly closed subset of X is LHC.

Proof If X is LHC, then every regularly closed subset is LHC. In particular every proper regularly closed subset is LHC.

Conversely, assume that every non-empty proper regularly closed subset of X is LHC. Let $B \subset X$ be a non-empty regularly closed subset of X. Note that $X = B \cup (X \setminus B) = B \cup cl_X(X \setminus B)$. Since B is a non-empty regularly closed subset and $int_X B \cap (X \setminus B) = \emptyset$, $X \setminus B$ is not dense in X.

Therefore, both B and $cl_X(X \setminus B)$ are proper non-empty regularly closed subsets of X and hence are LHC subsets. Thus X is a union of two LHC sets and hence is LHC.

Theorem 3.12 improves on Theorem 1 in [3]. It is stated here as Corollary 3.6.

Theorem 3.12 . If X is LHC and A, $B \subset X$ satisfy the conditions $cl_\theta^X(A) \cap cl_\theta^X(B) = \emptyset$, $cl_\theta^X(A) \cup cl_\theta^X(B) \neq X$, there are sets $V \in \Sigma_X(A)$, $W \in \Sigma_X(B)$ satisfying $V \cap W = \emptyset$.

Proof Neither $\mathrm{cl}_\theta^X(A) = X$ nor $\mathrm{cl}_\theta^X(A) = X$ holds, so a combination of Theorem 3.10 and Theorem 1 in [3] leads to the desired conclusion since $cl_\theta^{\kappa X}(A) = \mathrm{cl}_\theta^X(A)$.

Corollary 3.6. [3] *Subsets of an H-closed space with disjoint $\theta-$closures are separated by disjoint open sets.*

Using Theorem 3.8, the proof of the following theorem follows from Theorem 3.1 and is different from the proof of Theorem 3.8 of [7]. Moreover, the following Theorem extends Theorem 3.8 of [7].

Theorem 3.13. Let X be LHC and let $B \subsetneq X$ be open not dense in X. Then the following are equivalent:

1. *B is LHC.*
2. *$X \setminus B$ is θ -closed in X.*
3. *$cl_X B \setminus B$ is θ -closed in $cl_X B$.*
4. *$cl_X B \setminus B$ is θ-rigid in $cl_X B$.*
5. *$X \setminus B$ is θ-closed in any H-closed extension μX of X.*
6. *$X \setminus B$ is θ -rigid in any H-closed extension μX of X.*

Proof $(1) \Rightarrow (2)$. Let $x \in B$. There is an H-closed neighborhood V of x in B and hence V is a closed, in X, neighborhood of x in X, in view of Theorem 3.8. Also $V \cap (X \setminus B) = \emptyset$. Therefore $X \setminus B$ is θ -closed in X.

$(2) \Rightarrow (3)$. Follows from Lemma 3.1 of [7]

$(3) \Rightarrow (1)$. In view of Corollary 3.3, the (1) follows from the fact that $cl_X B$ is an H-closed extension of B.

Equivalence of (3) and (4) follows from Theorem 3.1.

Equivalence of (2) and (5) follows from $cl_\theta^{\mu X}(X \setminus B) \cap X = cl_\theta^X(X \setminus B) = (X \setminus B)$.

Equivalence of (5) and (6) follows from Theorem 3.1.

It is pointed out that the line of proof of $(1) \Rightarrow (2)$ is similar to that of the proof of the Proposition 3.2 of [7]. However, here the space X is LHC, not necessarily H-closed.

The following is established: *Let X be a Hausdorff space and let $A \subset X$. Then A is θ-rigid in X if and only if A is*

θ-rigid in κX if and only if A is θ-rigid in σX if and only if A is θ-rigid in some H-closed extension of X. [12].

Let θX be the compact and extremally disconnected space associated with X and let $\pi : \theta X \to \sigma X$ be defined by $\pi(\mathcal{U}) = \mathcal{U}$ for each free open ultrafilter $\mathcal{U}$ on X and $\pi(\mathcal{U}) = x$, where x is the unique convergent point of the fixed ultrafilter $\mathcal{U}$. Dickman and Porter proved that *if X is a Hausdorff space and $A \subset X$, then $\pi^{-1}(A)$ is compact if and only if A is θ-closed in κX* [1]. Thus, considering the fact that in an H-closed space, a set is θ-closed, if and only if it is θ-rigid, *if X is a Hausdorff space and $A \subset X$, then $\pi^{-1}(A)$ is compact if and only if A is θ-rigid in κX.*(Theorem 8, [12]). In view of the above Theorems, the following is a Corollary to Theorem 3.13:

Corollary 3.7. Let X be LHC and $B \subset X$ be open and not dense in X. Then B is LHC if and only if $\pi^{-1}(X \setminus B)$ is compact.

Proof Proof is immediate in view of the above discussion and the fact that $cl_\theta^{\kappa X}(X \setminus B) = cl_\theta^X(X \setminus B) = X \setminus B$.

Girou gave a characterization for an H-closed space to be compact using the concept of a rim θ-closed space [7]. A space X is *rim θ-closed* if it has a basis of open sets whose boundaries are θ-closed subsets of their closures [7]. We define a rim θ-rigid space and a rim u-rigid space as follows:

A space X is *rim θ-rigid* if it has a basis of open sets whose boundaries are θ-rigid subsets of its closures. A rim u-rigid space is defined similarly. A space X is *rim u-rigid* if it has a basis of open sets whose boundaries are u-rigid subsets of their closures. A θ-rigid subset of a space is u-rigid (Theorem 4 [4]). Hence a rim θ-rigid space is rim u-rigid and the converse is not necessarily true since there exist u-rigid subsets which are not θ-rigid [4]. Girou proved the following:

An H-closed space X is compact if and only if it is rim θ-closed if and only if every open subset is LHC if and only if it has an open base of LHC sets. ([7] Theorem 3.3)

In view of the above facts, the following Results follow from Theorem 3.13.

Corollary 3.8. Let X be LHC and $B \subseteq X$ be open and not dense in X. If B is LHC, then the following are true:

1. *$X \setminus B$ is u-rigid in any H-closed extension μX of X.*
2. *$cl_X B \setminus B$ is u-rigid in $cl_X B$.*

Proof Follows from the fact that a θ-rigid subset is u-rigid and Theorem 3.13 (4) and (6).

Following is an extension of Theorem 3.3 [7]:

Theorem 3.14. Let X be an LHC space. The following are equivalent for X:

1. *The space X is compact.*
2. *The space X is rim $\theta-$closed.*
3. *The space X is rim θ-rigid.*

Proof Follows from the fact that in an H-closed space family of θ-closed sets coincides with the family of θ-rigid sets.

Now the following Theorem is immediate.

Theorem 3.15. A compact Hausdorff space is rim u-rigid.

The next two Theorems show that the remainder of an LHC space in an H-closed extension as well as the boundary of a non-dense open subset of an LHC space can be written as the union of the u-closures of its singletons.

Theorem 3.16 If X is LHC, then

$$cl_u(\mu X \setminus X) = \bigcup_{\mu X \setminus X} cl_u(x)$$

for each H-closed extension μX of X.

Proof Since $\mu X \setminus X$ is θ-closed, $\mu X \setminus X$ is θ-rigid and hence it is u-rigid. Therefore

$$cl_u(\mu X \setminus X) = \bigcup_{\mu X \setminus X} cl_u(x)$$

follows from Theorem 5 of [4].

Theorem 3.17. Let X be an LHC space and let A be open, not dense in X and LHC. Then

$$cl_u(cl_X(A) \setminus A) = \bigcup_{cl_X(A) \setminus A} cl_u(x), \text{and}$$

$$cl_u(X \setminus A) = \bigcup_{X \setminus A} cl_u(x).$$

Proof Follows from Theorem 3.16 and Theorem 5 of [4].

Theorem 3.18. If X is Urysohn, the following statements are equivalent:

1. The space X is LHC;
2. $\kappa X \setminus X$ is u-rigid;
3. $\kappa X \setminus X$ is u-closed.

Proof (1)⇒(2) If X is LHC, $\kappa X \setminus X$ is θ-rigid and thus u-rigid [4].

(2)⇒(3) If $\kappa X \setminus X$ is u-rigid it is thus u-closed because $cl_u(\kappa X \setminus X) = \bigcup_{\kappa X \setminus X} cl_u\{x\}$.

(3)⇒(1)There exists $V \in \Lambda(x)$, where $\Lambda(x)$ represents the collection of open sets containing a closed neighborhood of x. Also, $V \cap (\kappa X \setminus X) = \emptyset$. There exists $W \in \sum(x), clW \subset V$. Consequently, $\kappa X \setminus X$ is θ-closed and hence, X is LHC.

Girou proved that if a space X is LHC and $B \subset X$, then B is LHC, if and only if (1) $cl_X B - B$ is a θ-closed subset of $cl_X B$, and (2) the set of nowhere dense points of B is LHC [7]. In the following theorem, it is shown that the two conditions for B to be LHC can be replaced by the condition that $cl_\theta B - B$ is θ-closed in $cl_\theta B$.

Theorem 3.19. Let X be LHC and $B \subseteq X$. Then B is LHC if and only if $cl_\theta B - B$ is θ-closed in $cl_\theta B$.

Proof Suppose that X is LHC and $B \subseteq X$ is LHC. Let $b \in B$. Since B is LHC, there exists an H-closed neighborhood V of b in B. Consider a $cl_\theta B$-open set U containing b. Then $U \cap V$ is an open subset of B containing b and $cl_{cl_\theta B}(U \cap V) \cap B = cl_B(U \cap V)$. Also $cl_B(U \cap V) \cap (cl_\theta B - B) = \emptyset$. Therefore, $cl_\theta B - B$ is θ-closed in $cl_\theta B$.

Suppose that $cl_\theta B - B$ is θ-closed in $cl_\theta B$. Let $b \in B$. Since X is LHC, $b \in X$, there is an open set G containing b and $cl_X G$ is H-closed. Also there exists a $cl_\theta B$ -open set U such that $cl_{cl_\theta B} U \cap (cl_\theta B - B) = \emptyset$. Without loss of generality, we assume that $U = G \cap cl_\theta B$. Now, $cl_{cl_\theta B} U = cl_X G \cap cl_\theta B$ and $cl_{cl_\theta B} U \subset B$, since $cl_{cl_\theta B} U \cap (cl_\theta B - B) = \emptyset$. Since $cl_\theta B$ is closed and $cl_X G$ is a regularly closed subset of X, $cl_{cl_\theta B} U$ is an H-closed neighborhood of b. Hence B is LHC.

We define a locally H-set as follows. A Hausdorff space X is *locally H-set* (LHS) if each point in X has an H-set neighborhood.

Theorem 3.20. A Hausdorff space X is LHC if and only if it is LHS.

Proof It will be shown that X is LHS iff $\kappa X \setminus X$ is θ-closed in κX. For the proof of the necessity, Choose $x \in X$ and choose an H-set neighborhood C of x in X. Then C is an H-set in κX, and is therefore closed in κX. Moreover, $C \cap (\kappa X \setminus X) = \emptyset$; so $\kappa X \setminus X$ is θ-closed in κX. For the proof of the sufficiency, choose an $x \in X$ and a $W \in \Sigma_X(x)$ such that $\mathrm{cl}_{\kappa X}(W) \subset X$. It follows that $\mathrm{cl}_{\kappa X}(W)$ is H- closed in X and $\mathrm{cl}_{\kappa X}(W) = \mathrm{cl}_X(W)$.

A Hausdorff space is called *locally regularly-closed* (LRC) if each point and an open set containing the point contains a regularly closed neighborhood of the point.

Theorem 3.21. A Hausdorff space X is LHC if and only if it is LRC.

Proof If H is an H-closed neighborhood of x, $\mathrm{cl}_X(\mathrm{int}(H)) \subseteq H$ is a regularly closed neighborhood of x. On the other hand, assume that X is LRC. Assume that every point in X and an open set containing the point contains a regularly closed set containing the point. Let $x \notin \kappa X \setminus X$, and let $V \in \Sigma_X(x)$.

Then there is a regularly closed set B such that $x \in B = cl_X U \subseteq cl_X(V)$, where U is an open subset of X. Then $\mathrm{cl}_{\kappa X}(U) \cap X = \mathrm{cl}_X(U)$. So if Γ is an open filter base on $\mathrm{cl}_{\kappa X}(U) \cap X$ then Γ is an open filter base on $\mathrm{cl}_{\kappa X}(U)$ and on $\mathrm{cl}_X(U)$. Hence $\emptyset \neq adh(\Gamma) \subset X$ since $\mathrm{cl}_{\kappa X}(U)$ is a regularly closed neighborhood of x in κX.

This shows that Γ is not free on X. Thus $\mathrm{cl}_{\kappa X}(U) \subset X$ and $\kappa X \setminus X$ is θ-closed in κX.

Ganster defined a space to be *strongly s-regular* if for any open set F and a point $x \in X - F$, there is a regularly closed set G such that $x \in G$ and $G \cap F = \emptyset$ [6].

It is known that every locally compact Hausdorff space is regular. The following analogous Theorem follows from the above Theorem:

Theorem 3.22. A Hausdorff LHC space is strongly s-regular.

Proof Immediate from Theorem 3.21 and Theorem 1 of [6].

Porter and Thomas showed that in a Hausdorff space a locally H-closed subspace is the intersection of an open set and a closed set ([19], Theorem 3.3). In the present article characterizations for an LHC space in terms of subsets are provided. Most of them are either in terms of regularly closed subsets or open subsets.

Following Corollaries of Theorem 3.22 give properties of closed subsets of a LHC space. The proofs of the following corollaries follow from Theorem 3.21 above and Theorem 3.1.12 of [16].

Corollary 3.9. If X is a Hausdorff LHC space then each closed set is an intersection of regularly open sets.

Corollary 3.10. If X is a Hausdorff LHC space, then each closed set is an intersection of semi-closed neighborhoods.

We end this section with the following characterizations of a LHC space X in terms of H-closed extensions of a subset B of X. Some of these results are stated in Theorem 3.13.

Theorem 3.23. Let $B \subseteq X$. Then the following statements are equivalent for a Hausdorff space X.

1. The space X is LHC;
2. $\mu B \setminus B$ is a θ-closed subset of μB for every H-closed extension μB of B;
3. $\mu B \setminus B$ is a θ-rigid subset of μB for every H-closed extension μB of B;
4. $\kappa B \setminus B$ is a θ-rigid subset of κB;
5. $\kappa B \setminus B$ is a θ-closed subset of κB.

Proof $(1) \Rightarrow (2)$. Let X be LHC, $B \subseteq X$ and $x \in B$. Then $x \in X$ and X is LHC. Therefore there is an open set $V \in \Sigma_X(x)$ such that $cl_X(V)$ is and H-closed subset of X. Now, $V \cap B$ is an open subset of B containing x and $cl_B(V) = cl_X V \cap B = cl_B(V \cap B)$. Also, $cl_{\mu X}(V) \cap X = cl_X(V)$. Similarly, $cl_{\mu B}(V \cap B) = cl_B(V \cap B)$, since $cl_B(V \cap B) = cl_{\mu B}(V \cap B) \cap B$. Therefore, $cl_{\mu B}(V \cap B) \cap (\mu B \setminus B) = \emptyset$. Hence $\mu B \setminus B$ is a θ-closed subset of μB.

$(2) \Rightarrow (3)$; $(4) \Rightarrow (5)$. These follow from [12].

$(3) \Rightarrow (4)$. Obvious.

$(5) \Rightarrow (1)$. Suppose that $\kappa B \setminus B$ is a θ -closed subset of κB and let $x \in X$. If $x \in B$, choose an open subset of $V \in \Sigma_X(x)$ such that $cl_{\kappa B}(V \cap B) \cap (\kappa B \setminus B) = \emptyset$. If $x \in X \setminus (\kappa B \setminus B)$, there is an open subset V of X containing x such that $cl_{\kappa B} V \cap (\kappa B \setminus B) = \emptyset$. However, for $V \subset X$, $cl_X(V) = cl_{\kappa X}(V)$ and hence is H-closed in X.

4. Examples

It is to be noted that any H-closed non-compact space will be a LHC space which is not locally compact. This is so, since every locally compact Hausdorff space is regular and a regular H-closed space is compact. Therefore, an H-closed space which is not compact is not locally compact. Hence, Following are Examples of LHC spaces which are not locally compact.

Example 4. 1 ([15] Example 3.6). Let $X = \{0\} \cup N \cup \{j + \frac{1}{n} : j,\ n \in N - \{1\}\}$ and define $V \subset X$ to be open if V satisfies the following properties:

(i) If $j \in (V \cap N) - \{1\}$, then $j + \frac{1}{n} \in V$ ultimately;
(ii) If $0 \in V$, then, ultimately, $j + \frac{1}{2n} \in V$ for all n;
(iii) If $1 \in V$, then, ultimately, $j + \frac{1}{2n+1} \in V$ for all n.

The above example is the classical example of a countable minimal Hausdorff space which is not compact. So, this is an example of a minimal Hausdorff space which is not locally compact.

Example 4. 2 Example 3.7). Let $Y = \{0\} \cup (N - \{1\}) \cup \{j + \frac{1}{2n} : j,\ n \in N - \{1\}\}$ with the subspace topology T from X in Example 4.1.

For this space, 0 is the only $\theta-$cluster point of $\{x_n\}$ defined by $x_n = n + 1$ but $\{x_n\} \not\to 0$. So, this space is not minimal Hausdorff and hence is not compact. The space is H-closed and hence is LHC, but not locally compact.

Girou, while considering those H-closed spaces in which every H-closed spaces are θ-closed, showed that such spaces are Urysohn [7]. The space in Example 4.1 is an H-closed space which is not Urysohn, since the points 0 and 1 do not have disjoint closed neighborhoods. Note that the space Y in Example 4. 2 is H-closed, but not θ-closed since $1 \in cl_\theta^X(Y)$ and $1 \notin Y$. It is also to be noted that the space in Example 4.1 is not rim θ-closed or rim u-closed, since the boundaries of the basic neighborhoods of points 0 and 1 are not θ-closed or u-closed. Since the space is H-closed, hence the space is not rim θ-rigid or rim u-rigid.

5. Conclusion

In the present article, LHC spaces are characterized using the remainders of the space in an H-closed extension. They are alo characterized using the subspaces and their extensions. Similar characterizations for LUC as well as LRC spaces are provided. Characterizing these classes of spaces using the remainders of the subspaces in their extensions provide characterizations of them in terms of their boundaries. As recent articles by the auhors, which are compiled in a monograph, 'A study of topological properties via adherence dominators', demonstrate that these results give necessary and sufficient conditions for the space to be compact [13]. Moreover, many results presented here provide generalizations of a number of existing results, which are pointed out throughout this article. In the present article it is shown that if X is a Hausdorff LHC space then each closed set is an intersection of regularly open sets as well as each closed set is an intersection of semi-closed neighborhoods. As stated earlier, Porter and Thomas showed that in a Hausdorff space a locally H-closed subspace is the intersection of an open set and a closed set [19].

Any result or definition from existing literature is given appropriate citation, as needed. However, those concepts which have been part of the literature on General Topology are not given reference citation, but we do not claim authorship of such concepts.

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