**American Journal of Applied Mathematics**
2021; 9(5): 156-164
http://www.sciencepublishinggroup.com/j/ajam
doi: 10.11648/j.ajam.20210905.11
ISSN: 2330-0043 (Print); ISSN: 2330-006X (Online)

# Method of Construction of the Stochastic Integral with Respect to Fractional Brownian Motion

**Diop Bou, Ba Demba Bocar*, Thioune Moussa**

Department Maths UFR SET, Iba Der Thiam University Thies, Thies, Senegal

## Email address:

dbba@univ-thies.sn (Ba D. Bocar)
*Corresponding author

## To cite this article:

Diop Bou, Ba Demba Bocar, Thioune Moussa. Method of Construction of the Stochastic Integral with Respect to Fractional Brownian Motion. *American Journal of Applied Mathematics.* Vol. 9, No. 5, 2021, pp. 156-164. doi: 10.11648/j.ajam.20210905.11

**Received:** July 2, 2021; **Accepted:** August 6, 2021; **Published:** September 3, 2021

**Abstract:** Since the pioneering work of Hurst, and Mandelbrot, the fractional brownian motions have played and increasingly important role in many fields of application such as hydrology, economics and telecommunications. For every value of the Hurst index $H \in (0,1)$ we define a stochastic integral with respect to fractional Brownian motion of index H. This process is called a (standard) fractional Brownian motion with Hurst parameter H. To simplify the presentation, it is always assumed that the fractional Brownian motion is 0 at t=0. If $H = 1/2$, then the corresponding fractional Brownian motion is the usual standard Brownian motion. If $1/2 < H < 1$, Fractional Brownian motion (FBM) is neither a finite variation nor a semi-martingale. Consequently, the standard Ito calculus is not available for stochastic integrals with respect to FBM as an integrator if $1/2 < H < 1$. The classic methods (Itô and Stiliege) are excluted. The most studied case is that where H is between 0 and 1/2. Several attempts to define the stochastic integral are made. But so far some difficulties subjust. We give in this paper, several construction methods. So for the construction, we will use other tools to deal with such situations.

**Keywords:** Wiener Integral, Fractional Brownian Motion, Martingale, Processus d'Ito

# 1. Introduction

Stochastic calculus is the study of random phenomena depending on the time. As such, it is an extension of probability theorie [10].

The heart of probabilistic tools lies in the stochastic calculus which is not nothing more than a differential calculus, but adapted to the trajectories of the processes stochastics that are not differentiable [18]. Differential calculus presents a theory of the integration of a stochastic (integrating) process with respect to another integrator, in order to solve stochastic differential equations which serve as mathematical models for systems involving two types of forces, one deterministic and the other random [15].

The stochastic integral $\int_0^t f(s, W_s)dW_s$ is called Wiener's integral for $f(s, W_s) = f(s)$. deterministic and is called ito integral in the general case of random $f(s, W_s)$. Our construction of the Wiener integral for f derivable functions can therefore be extended to any integrable square function f, keeping the stated properties. Indeed, for f of integrable square, let us take a sequence $(f_n)_n$ of regular functions converging quadratically towards $f : \int_0^t (f - f_n)^2(s)ds \to 0$. This sequence of functions is Cauchy in $L_2(\Omega, P)$ and by isometry, the sequence of random variables $(\int_0^t f_n(s)dW_s)_n$ is cauchy in $L_2(\Omega, P)$.

The Itô integral makes it possible to give meaning to most of the different equations stochastic ferentials from applied sciences [11]. But she has any time a limit: it only makes it possible to treat adapted integrals(or more pregradually measurable). Gaussian processes provide many examples of processes that are not semi-martingales among them fractional Brownian motion is widely used [12].

In 1993, Russia and P. Vallois [7] laid the first foundations for a stochastic calculus, generalizing the more classic ones of ito and tratonovich, one of the interests of which is that it makes it possible to give meaning to integrals against processses that are not not necessarily semi-martingales [14].

Gaussian processes provide many examples of processes that are not semimartingale. Among gaussian processes, fractional Brownian motion is widely used, its covariance function being particularity simple. This is why, in folliwing, this process will be used to test the general results that we will establish [4].

## 2. Russo and Vallois Method

In this part, we present new results to the theory generalized stochastic calculus developed by Russo and Vallois from 1993 wih makes it possible to give a meaning to integrals which are not necessarily semi-martingales.

### 2.1. Definition

Let $X = \{X_t, t \in \mathbb{R}_+\}$ et $Y = \{Y_t, t \in \mathbb{R}_+\}$ two continuous processes. We ask if the limit exists

$$\int_0^t Y_s d^- X_s = \lim_{\varepsilon \to 0} \ ucp \int_0^t Y_s \frac{(X_{s+\varepsilon} - X_s)}{\varepsilon} \mathrm{d}s \ \ Forward \ \ integral$$

$$\int_0^t Y_s d^+ X_s = \lim_{\varepsilon \to 0} \ ucp \int_0^t Y_{s+\varepsilon} \frac{(X_{s+\varepsilon} - X_s)}{\varepsilon} \mathrm{d}s \ \ backward \ \ integral$$

$$\int_0^t Y_s d^\circ X_s = \lim_{\varepsilon \to 0} \ ucp \int_0^t (Y_{s+\varepsilon}) \frac{(X_{s+\varepsilon} - X_s)}{2\varepsilon} \mathrm{d}s \ \ symmetrical \ \ integral$$

$$[X, X] = \lim_{\varepsilon \to 0} \ ucp \int_0^t \frac{(X_{s+\varepsilon} - X_s)^2}{\varepsilon} \mathrm{d}s \ \ quadratic \ \ variation$$

In these for definitions, ucp means uniform convergence in probability on compacts.

Reccall that a family $(X^\varepsilon)_\varepsilon$ converges in probability to x, uniformly on compacts if:

$$\forall T > 0, \forall \delta > 0 : \lim_{\varepsilon \to 0} P \left( \sup_{t \in [0,T]} |X_t^\varepsilon - X_t| > \delta \right) = 0$$

Indeed

We know that from hein lebesgue's theorem that if $f : I \subset \mathbb{R} \to \mathbb{R}$ a locally integrable function. So

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(u)du = f(x), \qquad \lambda \ \ p.p$$

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x}^{x+\varepsilon} f(u)du = f(x), \qquad \lambda \ \ p.p$$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(u)du = f(x), \qquad \lambda \ \ p.p$$

$\lambda$ denotes the Lebesgue measure.

And according to the stochastic version of Fubini's lemma, if M is a martingale continuous square integrable and if $H : \Omega \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a process bounded $B(\mathbb{R}^+)$ measurable, then, for all s, t $\geq 0$, we have

$$\int_0^s \left( \int_0^t H(u,v) dM_u \right) dv = \int_0^t \left( \int_0^s H(u,v) dv \right) dM_u$$

By applying these two theorems we will have

$$\int_0^t H(s) dX(s) = \int_0^t \left[ \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{s-\varepsilon}^s H(u) du \right] \mathrm{dX}(s)$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^t \left( \int_{s-\varepsilon}^s H(u) du \right) dX(s)$$

$$= \lim_{\varepsilon \to 0} \int_0^t \left( \int_u^{u+\varepsilon} dX(s) \right) H(u) du$$

$$= \lim_{\varepsilon \to 0} \int_0^t \frac{X(u+\varepsilon) - X(u)}{\varepsilon} H(u) du$$

Let $\varepsilon > 0$ and $t \geq 0$. If x and y are continuous stochastic processes, we pose

$$I^-(\varepsilon, t, X, dY) = \int_0^t X(s) \frac{Y(s+\varepsilon) - Y(s)}{\varepsilon} ds$$

and, if it exists $\int_0^t X d^- Y$ the limit in probability, when $\varepsilon$ approaches 0. This last quantity is then called forward integral.

These integrals allow us to write It$\hat{o}$'s formula the fractionnaire. Indeed when the processes x and y are semi-martingales and y is adapted in this case the integral of forward generalizes the stratonovich integral.

In [1] we prove the following result $\int_0^t f'(x_s) d^- x_s$ exists for any function $f \in C^2(\mathbb{R}, \mathbb{R})$ if and only if x admits a quadratic tion and in this case the following It$\hat{o}$ formula takes place

$$f(x_t) = f(x_0) + \int_0^t f'(x_s) d^- x_s + \frac{1}{2} \int_0^t f"(x_s) d[x,x]_s \quad t \in \mathbb{R} \tag{1}$$

When $X = B^H$.

We know that if $X = B^H$ an mbf of index $H \in (0,1)$. Then $B^H$ admits a quadratic variation if and only if $H \geq \frac{1}{2}$.

In this case it is worth the identity if $H = \frac{1}{2}$ and is identically zero if $H > \frac{1}{2}$. So (1) applies, for any function $f \in C^2(\mathbb{R},\mathbb{R})$. Under the same assumptions, $H \geq \frac{1}{2}$ and $f \in C^2(\mathbb{R},\mathbb{R})$, the symmetric integral $\int_0^t f'(x_s)d^\circ x_s$ exists and the formula Itô-Stratonovich takes places:

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_s^H)d^- B_s^H \qquad t \in \mathbb{R}_+ \qquad (2)$$

Because if $H > \frac{1}{2}, [B^H, B^H]$ therefore disappears the symmetric integral $\int_0^t f'(B_s^H)d^\circ B_s^H$ coincides with the forward integral $\int_0^t f'(B_s^H)d^- B_s^H$

If $H < \frac{1}{2}$, it is not possible to directly show the existence of the integral symmetrical. We can see that if $H \geq \frac{1}{4}$ fractional brownian motion adputs 4-variations. So if $H \geq \frac{1}{4}$ and if $f \in C^4(\mathbb{R},\mathbb{R})$ then $\int_0^t f'(B_s^H)d^\circ B_s^H$ exist and (2) takes place. This result is the basis of the first demonstration.

Indeed the authors do not directly show the existence of the integral of Stratonovich but introduce so-called order 3 integrals:

Let $X = \{X_t, t \in \mathbb{R}_+\}$ and $Y = \{Y_t, t \in \mathbb{R}_+\}$ are two continuous processes, we pose when the limit exists ,

$$\int_0^t Y_s d^{-(3)} X_s = \lim_{\varepsilon \to 0} \; prob \int_0^t Y_s \frac{(X_{s+\varepsilon} - X_s)^3}{\varepsilon} \mathrm{d}s$$

$$\int_0^t Y_s d^{\circ(3)} X_s = \lim_{\varepsilon \to 0} \; prob \int_0^t (Y_{s+\varepsilon} + Y_s) \frac{(X_{s+\varepsilon} + X_s)^3}{2\varepsilon} \mathrm{d}s$$

They prove that, for locally bounded function f, $\int_0^t f(B_s^H)d^{-(3)}B_s^H$ exist for $H \geq \frac{1}{4}$.

More precisely, if H $> \frac{1}{4}$, $\int_0^t f(B_s^H)d^{-(3)}B_s^H$ exist and is equal to 0 and if $H = \frac{1}{4}$, the integral $\int_0^t f(B_s^{\frac{1}{4}})d^{-(3)}B_s^{\frac{1}{4}}$ exists but is not worth 0 in general.

They finally deduce that the Itô-Stratonovich formula

$$f(B_t^H) = f(B_0^H) + \int_0^t f'(B_s^H)d^\circ B_s^H, \quad t \in \mathbb{R}_+$$

takes place for any function $f \in C^4(\mathbb{R})$ if $H \geq \frac{1}{4}$.

We know that now now that Itô's formula exists if $H > \frac{1}{2}$ and the formula of Itô-Stratonovich occurs if $H > \frac{1}{4}$[**5**].

A fundamental question arises to know for which values of H the formula d'Itô and Stratonovich exist. The authors show with the help of Ivan Nourdin that this set is $]\frac{1}{6}, 1[$.

Indeed they first showed that the Itô-Stratonovich formula cannot have place $H < \frac{1}{2}$. We have for $s \geq 0$ et $\varepsilon > 0$ :

$$(B_{s+\varepsilon}^H)^3 = (B_s^H)^3 + 3\frac{(B_{s+\varepsilon}^H)^2 + (B_s^H)^2}{2}(B_{s+\varepsilon}^H - B_s^H) - \frac{(B_{s+\varepsilon}^H - B_s^H)^3}{2}$$

By integrating for $s \in [0;t]$ and dividing by $\varepsilon$, we get:

$$\frac{1}{\varepsilon}\int_0^t [(B_{s+\varepsilon}^H)^3 - (B_s^H)^3]\mathrm{d}s = \frac{1}{\varepsilon}\int_t^{t+\varepsilon} (B_{s+\varepsilon}^H)^3 \mathrm{d}s - \frac{1}{\varepsilon}\int_0^\varepsilon (B_s^H)^3 \mathrm{d}s$$

$$= \frac{3}{\varepsilon}\int_0^t \frac{(B_{s+\varepsilon}^H)^2 + (B_s^H)^2}{2}(B_{s+\varepsilon}^H - B_s^H)\mathrm{d}s - \frac{1}{2\varepsilon}\int_0^t (B_{s+\varepsilon}^H - B_s^H)^3 \mathrm{d}s$$

By doing $\varepsilon \to 0$, we deduce that the symmetric integral $\int_0^t (B_s^H)^2 d^\circ B_s^H$ exist if and only if its cubic variation $[B^H, B^H, B^H]$ exists and in this case:

$$(B_t^H)^3 = (B_0^H)^3 + 3\int_0^t (B_s^H)^2 d^\circ B_s^H - \frac{[B^H, B^H, B^H]_t}{2}.$$

Therefore, for the Ito-Stratonovich formula to take place for the function $f(x) = x^3$, it is nacessary that the symmetric integral $\int_0^t (B_s^H)^2 d^\circ B_s^H$ exist and therefore that the cubic variation $[B^H, B^H, B^H]$ exists[**9**].

Indeed By setting $f(x) = x^3$ we have

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)d^- X_s + \frac{1}{2}\int_0^t f"(X_s)d[X,X]_s$$

$$f(X_t) = f(X_0) + 3\int_0^t X^2 d^- X_s + 3\int_0^t X d[X,X]_s$$

$$f(B_t^H) = f(B_0^H) + 3\int_0^t [B_s^H]^2 d^- B_s^H + 3\int_0^t B_s^H d[B_s^H, B_s^H]_s$$

$$f(B_t^H) = f(B_0^H) + 3\int_0^t [B_s^H]^2 d^- B_s^H - \frac{[B_s^H, B_s^H, B_s^H]}{2}$$

Now Nourdin with Gradinaru [8] showed that if $m \geq 3$ an odd integer and suppose that $H \in [0; \frac{1}{2}]$ we have

$$\left\{ \frac{1}{\sqrt{\varepsilon}} \int_0^t \left( \frac{B_{s+\varepsilon}^H - B_s^H}{\varepsilon^H} \right)^m \mathrm{d}s : t \geq 0 \right\} \to_{(loi)} \left\{ \sqrt{c_{m,H}} \beta_t : t \geq 0 \right\}, \quad \varepsilon \to 0.$$

Here $\{\beta_t : t \geq 0\}$ denotes a standart Brownian motion from 0 and the positive constant $c_{m,H} = 2\sum_{k=1}^{m} \frac{a_{k,n}^2}{k!} \int_0^\infty \left[ (x+1)^{2H} + |x-1|^{2H} - 2x^{2H} \right]^k$

We therefore deduce that the cubic variation $[B^H, B^H, B^H]$ does not exist for $H < \frac{1}{6}$.

Indeed, suppose that $\frac{1}{\varepsilon}\int_0^t (B_{u+\varepsilon}^H - B_u^H)^3 \mathrm{d}u$ of converges in probability, when $\varepsilon \to 0$, to a random variable z, we deduce that

$$\varepsilon^{\frac{1}{2} - 3H} \frac{1}{\varepsilon} \int_0^t (B_{u+\varepsilon}^H - B_u^H)^3 \mathrm{d}u \to_{loi} 0 \;\; of \; \varepsilon \to 0$$

But this quantity is also equal to $\frac{1}{\sqrt{\varepsilon}} \int_0^t \left( \frac{B_{u+\varepsilon}^H - B_u^H}{\varepsilon^H} \right)^3 \mathrm{d}u$, and converges law, from the above to $\sqrt{c_{3,H} t} N$. We get a contradiction.

Therefore, the Ito Stratonovich formula cannot take place if $H < \frac{1}{6}$.

What happens for $H > \frac{1}{6}$

Nourdin, Gradinaru, Russo and Vallois [2] have shown that the symmetrical integral order 3 that $\int_0^t f(B_s^H) d^{\circ(3)} B_s^H$ exists for any function sufficiently regular if $H > \frac{1}{6}$ and that in this case, it is zero.

Nourdin also showed that the symmetric integral of order 5 $\int_0^t f(B_s^H) d^{\circ(5)} B_s^H$ exists for any sufficienty regular function if $H > \frac{1}{10}$ and that in this case, she is bad.

Using Taylor's formula for $f \in C^6(\mathbb{R}, \mathbb{R})$:

$$f(b) = f(a) + \frac{f'(b) + f'(a)}{2}(b-a) - \frac{f^3(b) + f^3(a)}{12}(b-a)^3 + \frac{f^5(b) + f^5(a)}{120}(b-a)^5 + \circ((b-a)^6) \quad a, b \in \mathbb{R}$$

by setting $a = B_s^H$ , $b = B_{s+\varepsilon}^H$, by integrating for $s \in [0, t]$ by dividing by $\varepsilon$ and by making $\varepsilon \to 0$ that, if $H > \frac{1}{6}$

$$f(B_t^H) = f(0) + \int_0^t f'(B_s^H) d^\circ B_s^H - \frac{1}{12}\int_0^t f^{(3)}(B_s^H) d^{\circ(3)} B_s^H + \frac{1}{120}\int_0^t f^{(5)}(B_s^H) d^{\circ(5)} B_s^H$$

Since the symmetric integrals of order 3 and 5 are zero, the proceding formula dente is in fact reduced to the Stratonovich integral.

Consequently, Ito Stratonovich formula holds for any function f suffisufficiently regular (precisely class $C^6$) if $H > \frac{1}{2}$.

If $H < \frac{1}{6}$ then the 3-variations does not exist so we have to define a new class of integrals.

Let x be a continuous process. An integer $m \geq 1$ is a measure of probability $\mu$ on [0,1] have defined $\mu$ integral of order m of f(x) by :

$$\int_0^t f(X_u) d^{\mu,m} X_u = \lim_{\varepsilon \to 0} prob \frac{1}{\varepsilon} \int_0^t du (X_{u+\varepsilon} - X_u)^m \times \int_0^1 f(X_u + \alpha(X_{u+\varepsilon} - X_u)) \mu \mathrm{d}\alpha$$

Let n and be two strictly positive integers. Suppose $\mu$ is a measure of symmetric probability on [0,1] such that $m_{2j} = \int_0^1 \alpha^{2j} \mathrm{d}(\alpha) = \frac{1}{2j+1}$ pour j = 1,$\cdots$ l-1

If $f \in C^{2n}(\mathrm{R})$ and if x is a continuous process admitting one (2n)-variations, provided that all the integrals in play exist, we have the Ito formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_u) \mathrm{d}^{\mu,1} \mathrm{X_u} + \sum_{j=l}^{n-1} K_{l,j}^u \int_0^t f^{2j+1} X_u d^{\delta_{1/2}, 2j+1}$$

Or the sum is zero by convention if $l > n-1$ here the $K^{\mu}_{(l,j)}$ are constants universal. Let us now explore the existence and non-existence of $\int_0^t g(X_u)\mathrm{d}^{\mu,\mathrm{m}}\mathrm{X_u}$

If $X = B^H$ an mbf of Hurst index $H \in (0,1)$ and $\mu$ a probability measure on [0,1]. We denote by $\mu_{2n}$ the moment of order 2n of the centered normal law and reduced. The following theorem gives the complete description of the different cases possibles.

### *2.2. Theorem 1*

Let $m \geq 2$ be an integer and $\mu$ be a probability measure on [0,1]

1) Suppose that m = 2n is even and g is a locally bounded function. So:

a) If $2nH \geqslant 1$ then $\int_0^t g(B_u^H)\mathrm{d}^{\mu,2\mathrm{n}}\mathrm{B_u^H}$ exist and

$$\int_0^t g(B_u^H)\mathrm{d}^{\alpha,2\mathrm{n}}\mathrm{B_u^H} = \int_0^t g(B_u^H)\mathrm{d}[\mathrm{B^H}]_{\mathrm{u}}^{(2\mathrm{n})} = \mu_{2n}$$

$$\begin{cases} \int_0^t g(B_u^H)\mathrm{du} & if 2nH = 1 \\ 0 & if\ 2nH > 1; \end{cases}$$

b) If $2nH < 1$ then $\int_0^t g(B_u^H)\mathrm{d}^{\mu,2\mathrm{n}}\mathrm{B_u^H}$ en general does not exist.

2) Suppose that m =2n+1 is odd, that g is a function of class $C^{2n+1}$ and that $\mu$ is symmetric . So:

a) If $(2n+1)H > \frac{1}{2}$ so $\int_0^t g(B_u^H)\mathrm{d}^{\mu,2\mathrm{n}+1}\mathrm{B_u^H}$ exists and is canceled.

b) If $(2n+1)H < \frac{1}{2}$ so $\int_0^t g(B_u^H)\mathrm{d}^{\mu,2\mathrm{n}+1}\mathrm{B_u^H}$ in general does not exist. For **Proof** see [**11**]

## 3. Using the Integral and the Fractional Derivative

### *3.1. Recall*

Riemann-Liouville fractional integrals on $\mathbb{R}$ are defined by

$$I_+^{\alpha} = \frac{1}{\Gamma\alpha}\int_{-\infty}^{x} f(x)(x-t)^{\alpha-1}\mathrm{d}t$$

$$I_+^{\alpha} = \frac{1}{\Gamma\alpha}\int_{x}^{\infty} f(x)(x-t)^{\alpha-1}\mathrm{d}t$$

Let $H \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$ and $\alpha = H - \frac{1}{2}$, forall $t \in \mathbb{R}$ we have the equality

$$I_-^{\alpha}1_{(0,t)}(s) = \frac{1}{\Gamma}\int_x^{\infty} 1_{(0,t)}(u)(u-x)^{\alpha-1}$$

Indeed if $H \in (\frac{1}{2},1)$

$$I_-^{\alpha}1_{(0,t)}(x) = \frac{1}{\Gamma(\alpha)}\int_0^t (u-x)^{\alpha-1}\mathrm{d}u = \frac{1}{\Gamma(\alpha+1)}[(t-x)^{\alpha} - (-x)^{\alpha}]$$

Using the antiderivative of $(u-x)^{\alpha-1}$

In the same way have

$$I_+^{\alpha}1_{a,b}(s) = \frac{1}{\Gamma(\alpha+1)}((b-x)^{+}\alpha - (a-x)_+^{\alpha})$$

Let $f \in L^1(\mathbb{R})$. The Fourier transform of f defined as.

$$(\mathcal{F}f)(s) = \widehat{f}(x) = \int_{\mathbb{R}} e^{ixt}f(t)dt$$

[2].

We say that f is a step-by-step function, or an elementary function, if there exists a finite number of points $t_k \in \mathbb{R}, 0 \leq$

$k \leq n-1$, and $a_k \in \mathbb{R}$, such that

$$f(t) = \sum_{k=1}^{\infty} a_k 1_{[t_{k-1},t_k[}(t)$$

For $H \in (0,1)$ we introduce the set

$$\mathcal{F}_H = \{f \in L_2(\mathbb{R}), f : \mathbb{R} \to \mathbb{R}, \int_{\mathbb{R}} \mid \widehat{f(x)} \mid^2 |x|^{-2\alpha} \mathrm{d}x\}$$

Let $f \in \mathcal{F}_H$. Then there exists a sequence of functions $f_n$ such that

$$\|f - f_n\|_{\mathcal{F}n} \to 0,\ n \to \infty$$

Indeed if $f \in L^2(\mathbb{R}^+)$, there exists a sequence $f_n$ of staircase functions which converges in $L^2(\mathbb{R}_+)$, to f that is to say

$$\int_0^{\infty} \mid f_n - f \mid^2 \mathrm{d}x \to 0 \ \ if\ n \to \infty$$

In this case, the sequence $f_n$ from Cauchy in $L^2(\mathbb{R}_+)$

### 3.2. Theorem 2

For $H \in (0,1)$, the set $\mathcal{F}_H$ is a product linear space noted:

$$(f,g)_{\mathcal{F}_H} = \int_{\mathbb{R}} \widehat{f(x)}\overline{\widehat{g(x)}}|x|^{-2\alpha}\mathrm{d}x, \qquad \alpha = H - \frac{1}{2}$$

Moreover all basic functions belong to $\mathcal{F}_H$. *Proof* Indeed if a and b are finite we have

$$\int_{\mathbb{R}} |\widehat{1_{a,b}(x)}|^2 |x|^{-2\alpha}\mathrm{d}x = \int_{\mathbb{R}} |e^{ixb} - e^{ixa}|^2 |x|^{-2-2\alpha}\mathrm{d}x$$

is equivalent to the convergent $\int |x|^{-2\alpha-2}\mathrm{d}x$ in the neighborhood of $\pm\infty$ and is equivalent to the convergent $\int |x|^{-2-2\alpha}\mathrm{d}x$ in the neighborhood of 0.

We deduce that the step function belongs to $\mathcal{F}_H$

To obtain the representation of mbf in terms of an indicator functions, you can use fractional integrals and fractional derivatives first for $H \in (\frac{1}{2},1)$, we can use fractional integrals and for $H \in (0,\frac{1}{2})$ we use fractional derivates [3].

Indeed the stochastic integral with respect to the mbf has been defined as principalment for deterministic or linear integrals.

In the general case and particularty it is more complicated to establish such an integral, since the regularity of the trajectories of mbf varies with the Hurst parameter. In the general case and particularty when $H > \frac{1}{2}$, the trajectories of the mbf are essentially $\alpha$-Hölder continuous for $\alpha < H$, therefore, a trajectory stochastic integral approach is also effective in the same way as the presented by Young.

In the general case, when $H < \frac{1}{2}$, the trajectories of the mbf become more rupigs and therefore the trajectory approa ch for the stochastic integral is not consistent and therefore unnecessary. For this reason other definitions of in stochastic tegrals were introduced. Most notable is the integration of the divergent type (or the integral of Skorohod) which is based on the idea of calculus by Malliavin [13].

We define the operator $M_{\pm}^H f = C_H^{(3)} I_{\pm}^{\alpha} f,\ H \in (\frac{1}{2},1)$ and $M_{\pm}^H f = f$, if $H = \frac{1}{2}$ and $M_{\pm}^H = C_H^3 D_{\pm}^{-(H-\frac{1}{2})} f$ if $H \in (0,\frac{1}{2})$ where $C_H^{(3)} = C_H^{(2)}\Gamma(H+\frac{1}{2})$

So we can deduce $B_t^H = \int_{\mathbb{R}} (M_-^H) 1_{(0,t)}(s) dw_s$ is a browmovement mine fractional normalized.

Indeed:

$$B_t^H = \int_{\mathbb{R}} M_-^H 1_{0,t}(s)\mathrm{dw_s}$$

$$B_t^H = \int_{\mathbb{R}} C_H^{(3)} I_{\pm}^{\alpha} 1_{0,t}(s)\mathrm{dw_s}$$

$$= C_H^{(3)} \int_0^t K_H(t,u)\mathrm{dw_s}$$

### 3.3. Definition 1

We now return to an arbitrary full space $(\Omega, \mathcal{F}, P)$.

We denote by $\varepsilon$ the space of stepped functions. We can define the Wiener integral staged with respect to fractional Brownian motions as follow: for a mbf $(B_t^H)_{t\leq 0}$ one defines the integral of Wiener compared to the mbf for $f \in \varepsilon$ by:

$$I^H = \int_T f(u)\mathrm{d}B_u^H = \sum_{k=1}^{n} f_k (B_{u_{k+1}}^H - B_{u_k}^H), \quad T = [0,t]$$

and f(u) = $\sum_{k=1}^n f_k 1_{[u_k,u_{k+1}]},\ \ u \in [0,t]$

We extend the application $I^H$ in a space of integrants which is a space provided wich a scalar product, this space is denoted $L_2^H$ Consider the space $L_2^H(\mathbb{R}) = \langle f : M_-^H f \in L_2(\mathbb{R})\rangle$ provided wich the standard

$$\|f\|_{L_2^H(\mathbb{R})} = \|M_-^H f\|_{L_2(\mathbb{R})}$$

### 3.4. Definition 2

The Wiener integral with respect to mbf is the isometric map $I^H$ definished as:

$$I^H : L_2^H \to \overline{S_P(B_H)}$$

$$f \to I^H(f) = X$$

So we can define

$$S_p(B^H) = \{X, I^H(f_n) \to X, f_n(x) \subset \varepsilon\}$$

we associate x with a sequence of stepped functions $(f_n(x))_{n\in N}$, equivalent

$I^H(f_n) \to X$. In addition we can write $\int_T f_x(t)\mathrm{d}B_t^H$, where $f_x$ is an element of equivalence classes.

For $H > \frac{1}{2}$

To make sense of a $I_H(f) = \int_{\mathbb{R}} f(s)dB_s^H$ we construct interest classes integrating of deterministic functions.

First for $f \in \varepsilon$ where $\varepsilon$ is the set of staircase functions on $\mathbb{R}$.

In a naturel way we have

$$I_H(f) = \int_{\mathbb{R}} f(s)\mathrm{d}B_s^H = \sum_{k=1}^{n} a_k(B_{t_K}^H - B_{t_{K-1}}^H)$$

The we extend this definition to a class of integrant larger than $\varepsilon$. For that we consider the representation of mbf on $\mathbb{R}$ given by:

$B_t^H = \int_{\mathbb{R}} M_-^H 1_{(0,1)} \mathrm{d}w_s$ thus for all functions f and g $\in \varepsilon$ we have

$$\int_{\mathbb{R}} f(s)\mathrm{d}B_s^H = \int_{\mathbb{R}} (M_-^H f)(s)\mathrm{d}w_s$$

And using the well-know isometric property of the Wiener integral of the standard Brownian motion we write

$$E\left(\int_{\mathbb{R}} f(s)dB_s^H \int_{\mathbb{R}} g(s)dB_s^H\right) = E\left(\int_{\mathbb{R}} (M_-^H f)(s)dw_s \int_{\mathbb{R}} (M_-^H g)(s)dw_s\right) = \int_{\mathbb{R}} (M_-^H f)(s)(M_-^H g)(s)ds$$

We put $L_2^H = \left\{ f, \int_{\mathbb{R}} (M_-^H f)^2(s)ds < \infty \right\}$

A Gaussian space is a closed vector subspace of $L_2(\Omega)$ composed of one-dimensional centered Gaussian random variables.

Let $B_t^H$ a centered one-dimensional Gaussian process. Then, $\overline{vect(B_{t_i}^H, t \in [0,T])}^{L_2(\Omega)}$ is the Gaussian space generated by the w process.

Let $\overline{S_p(B_H)} = \overline{vect(B_{t_i}^H, t \in [0,T])}^{L_2(\Omega)}$ the space created by the closure in $L_2(\Omega)$ of all linear combinations of mbf increments on T. Since $\overline{S_p(B^H)}$ is a complete linear space then the class of interest grant must be isometric to the consequently complete Gaussian space. Gold for $H \in ]\frac{1}{2}, 1[$ the integrant class $L_2^H$ is not complete but we can find isometric eigenspaces $\overline{vect(B_{t_i}^H, t \in [0,T])}^{L_2(\Omega)}$ to $L_2^H$.

### 3.5. Theorem 3

Space $L_2^H$ is not complete for $H \in (\frac{1}{2}, 1)$

*Proof* The operator $M_-^H : L_2^H(\mathbb{R}) \to L_2(\mathbb{R})$ is isometric. Thus, $L_2^H$ can be identified relied on its image in $L_2(\mathbb{R})$. From the fact that for $H \in (\frac{1}{2}, 1)$.

Operator $M_-^H$ coincides with

$D(I_-^{-\alpha}) = D\left(D_-^{\alpha} = U_{p \geq 1} I_-^{\alpha}(L_p(\mathbb{R}))\right)$

Therefore quent, the image in $M_-^H(L_2^H(\mathbb{R}))$ is dense and is not complete. Despite the fact the $L_2^H$ is not complete for $H \in (\frac{1}{2}, 1)$, due to the theorem following one can extend the map $I^H$ on $\varepsilon$ to the functions of $L_2^H$, provided with scalar product therefore

$$\langle f, g \rangle_{L_2(\mathbb{R})} = \int_{\mathbb{R}} (M_-^H f)(s)(M_-^H g)(s)\mathrm{d}s$$

### 3.6. Theorem 4

For $0 < H < 1$, the set of linear operators generated by $\left\{M_-^H 1_(u,v), u, v \in \mathbb{R}\right\}$ is dense in $L_2(\mathbb{R})$.

*Proof* i) Let $H \in (\frac{1}{2}, 1)$ (for $H = \frac{1}{2}$ the assertion is evident). Since $(b+x)^{-\alpha} - x^{-\alpha} \sim cx^{-\frac{1}{2}-H}$ as $x \to \infty$, we have that the function $(b-x)_+^{-\alpha} - (-x)_+^{-\alpha} \in L_{\frac{1}{H}}(\mathbb{R})$.

Therefore $1_{(a,b)} = M_-^{\alpha} g \in I_-^{\alpha}(L_{\frac{1}{H}}(\mathbb{R}))$, and this is true also for step function. Since the class of step functions is dense in $L_{\frac{1}{H}}(\mathbb{R})$, it follow that $I_-^{\alpha}(L_{\frac{1}{H}}(\mathbb{R}))$ is dense in $L_2(\mathbb{R})$.

Let $h \in I_-^{\alpha}(L_{\frac{1}{H}}(\mathbb{R}))$, $h = M_-^H g, g \in L_{\frac{1}{H}}(\mathbb{R})$. Then there exists the sequence of step functions $g_n \to g \in L_{\frac{1}{H}}(\mathbb{R})$. From the Hardy Littlewood theorem it follow that

$$\|M_-^H g_n - h\|_{L_2(\mathbb{R})} \leq c\|g_n - g\|_{L_{\frac{1}{H}}(R)} \to 0, n \to \infty$$

So, the linear span of $\{M_-^H 1_{(u,v)}, u, v \in \mathbb{R}\}$ is dense in $I_-^{\alpha}(L_{\frac{1}{H}}(\mathbb{R}))$, and therefore it is dense in $L_2(\mathbb{R})$.

(ii) Let $H \in (0, \frac{1}{2})$. Due to the Parceval identity, it is sufficient to prove that the linear span of the function $\widehat{M_-^H 1_{(a,b)}}$ is dense in $L_2(\mathbb{R})$.

Since the set of functions stepped $\varepsilon$ is dense in $L_2^H$ can extend the application

$$I_H(f) : \varepsilon \to \overline{S_p(B_H)}$$

Since the extention $I_H(f)$ is linear and preserves the dot product, we can say that $L_2^H$ is isometric to a subspace of $\overline{S_p(B_H)}$. We can therefore approximate mer any functions f of $L_2^H$ by a stepped function $f_n$ so $M_-^H f_n \to M_-^H f$ so

$$\begin{aligned}
\lim_{n\to\infty} \int_{\mathbb{R}} f_n(s)\mathrm{d}B_s^H &= \lim_{n\to\infty} \int_{\mathbb{R}} M_-^H f_n(s)\mathrm{d}w_s \\
&= \int_{\mathbb{R}} M_-^H f(s)\mathrm{d}w_s \\
&= \int_{\mathbb{R}} f(s)\mathrm{d}B_s^H \\
&= I_H(f)
\end{aligned}$$

where convergence takes place in $L_2(\Omega)$.

On the other hand $E|I_H(f)|^2 = \int_{\mathbb{R}} |M_-^H f(s)|^2 \mathrm{d}s \quad for\ f \in L_2^H$.

For $H < \frac{1}{2}$

We always consider first the function $f : \mathbb{R} \to \mathbb{R}$ defined by:

$$f(t)=\sum_{k=1}^{n} a_k 1_{[t_{k-1},t_k]}(t)\ a_k\in\mathbb{R}\ \ and\ t_{k-1}<t_k,\ k=1,\ldots,n$$

In a natural way we have

$$I_H(f)=\int_{\mathbb{R}} f(s)\mathrm{d}B_s^H=\sum_{k=1}^{n} a_k(B_{t_k}-B_{t_{k-1}})$$

Then we extend this definition to a class of integrant larger than $\varepsilon$. For that we consider the representation of mbf on $\mathbb{R}$ given by:
$B_t^H=\int_{\mathbb{R}} M_-^H 1_{(0,t)}(s)\mathrm{d}w_s$. Thus for all functions f and g $\in\varepsilon$ we have

$$\int_{\mathbb{R}} f(s)\mathrm{d}B_s^H=\int_{\mathbb{R}}(M_-^H)f(s)\mathrm{d}w_s$$

and using the well-know isometric property of the Wiener integral of the standard Brownian motion we write

$$E\left(\int_{\mathbb{R}} f(s)\mathrm{d}B_s^H\int_{\mathbb{R}} g(s)\mathrm{d}B_s^H\right)=\int_{\mathbb{R}}\left(M_-^H f\right)(s)\left(M_-^H g\right)(s)\mathrm{d}s$$

We put $L_2^H(\mathbb{R})=\left\{f,\int_{\mathbb{R}}\left(M_-^H f\right)^2(s)\mathrm{d}s<\infty\right\}$ with $\left(M_-^H f\right)=\left(C_H^{(3)}D_-^{\frac{1}{2}-H}f\right)$

For H$<\frac{1}{2}$ space $L_2^H(\mathbb{R})$ coincides with space $\Lambda=\left\{f,\exists\phi f\in L_2(\mathbb{R}),f=I_-^{-\alpha}\phi f\right\}$ for H$\in(0,\frac{1}{2})$. The set of staircase functions $\varepsilon$ is dense in space linear $L_2^H(\mathbb{R})=\Lambda$ with the dot product

$$\langle f,g\rangle_{L_2^H}(\mathbb{R})=\int_{\mathbb{R}}(M_-^H f)(s)(M_-^H g)(s)\mathrm{d}s=\left(C_H^3\right)^2\int_{\mathbb{R}}(D_-^{\frac{1}{2}-H}f)(s)(D_-^{\frac{1}{2}-H}g)(s)\mathrm{d}s$$

### ***3.7. Theorem 5***

For $H<\frac{1}{2}$, provided with the scalar product the space $\Lambda$ is a complete space. *Proof* Let $\{f_n\}_{n\geq 1}$ a Cauchy sequence in $\Lambda$ then there exists a sequence $\phi_{f_n}$ which is Cauchy in $L_2(\mathbb{R})$ such that $\phi_{f_n}\to\phi$.

If $f(u)=(I_-^{-\alpha}\phi)(u)$ then $f_n\to f$ in $\Lambda$ since $\phi(f_n)\to\phi$ in $L_2^H(\mathbb{R})$ is complete and is equal to the closure of the space of staircase functions $\varepsilon$ under the standard $\parallel M_-^H\parallel_{L_2(\mathbb{R})}$. By the isometric relation

$$\parallel I_H(f)\parallel_{L_2(\mathbb{R})}=\sum_{i=1,k=1} a_i a_k\int_{\mathbb{R}} M_-^H 1_{[t_{k-1},t_k]}(x)M_-^H 1_{[t_{k-1},t_k]}(x)\mathrm{d}x$$

Where there is a unique extension of the Wiener fractional integral for staircase functions to spaces $L_2^H(\mathbb{R})=\wedge$.

Thus for any function $f\in L_2^H(\mathbb{R})$ there exists a staircase function $f_n$ such that we define the Wiener integral by

$$I_H(f)=\int_{\mathbb{R}} f(s)\mathrm{d}B_s^H=\lim_{n\to\infty}\int_{\mathbb{R}} f_n(s)\mathrm{d}B_s^H\ \ in\ L_2(\mathbb{R})$$

# 4. Conclusion

In this exposed, the work which was to give meaning, to the integral of determined functions nists (called Wieners integral) with respect to fractional Brownian motion is ahieved in a part by constructing classes of integrants for these deterministic functions which allows us to define the integral [16].

The construction of these classes to make an integration was necessary since the movement fractional Brownian event is a general case of standard brownian motion. This while it should be noted that fractional brownian motion loses certain properties such as semimartingality, the property of markovs which allowed the construction of a integral with respect to standart brownian motion. It is not also quadratic finished, il was then necessary to develop a new stochastic calculus and therefore new methods whose applications can be found in such varies fiels as image ment, banking and insurance in finance, modeling in physics, telecommunication ... .

The representation of fractional brownian motion over R and over an inteval [0,T] thanks to the elements of fractional calculus allowed us to construct classes of interest grants for $H>\frac{1}{2}$ and classes of integrants for $H<\frac{1}{2}$ in the case where $H>\frac{1}{2}$ the space is not complete, we were able to buld clean subspace to make integration [1].

In the literature, there is only one other construction (as far as we know) of integral against fractional brownian motion allowing to treat, as here, the case of all Hurst indices $H\in(0,1)$.This is in fact a very difficult problem, which has only recently been solved.

## References

[1] Alos, E., Leon, J. L, Nualart, D (2001). Stratonovich calculus for fractional Brownian motion with Hurst parameter less than 1/2. Taiwanese J. Math. 5, 609-632.

[2] Ba Demba Bocar: On the fractional Brownien motion: Hausdorf dimension and Fourier expansion international journal of advances in applied mathematical and mechanics vol 5 pp 53-59 (2017).

[3] Ba Demba Bocar: Fractional operators and Applications to fractional martingal international journal of advances in applied mathematical and mechanics vol 5 (2018).

[4] Bickel, P. J. Doksum, K. A. (1977) Mathematical Statistics. Prentice Hall, Inc.

[5] D. Nualart (2003): Stochastic calculs with respect to the fractional Brownian motion and applications. Contemporaty Mathematics 336, 3-39.

[6] F. Russo and P. Vallois (2005): Elements of stochastic calculus via regularisation.

[7] F. Russo. Vallois P. The generalized covariation process and I$t\hat{o}$ formula Stochastic Processes and their application 59, 81-104, 1995.

[8] Gradinaru, M, Nourdin, I. (2003). Stochastic volatility: approximation and goodness-of-fit test. Prepint IECN 2003-53.

[9] Gradinaru, M. Nourdin, I (2003) Approximation at first and second order of m. order integrals of the fractional motion and of certain semi martingales. J Probab. 8 no 18.

[10] Gradinaru, M. Nourdin, I Russo, F. Vallois, P. (2004) M-order integrals and Ito's formula for non-semimartingal processus: the case of a frational Brownien notion Withany Hurst index.

[11] H Doss (1977): Links between stochastic and ordinary differential equations. Ann. Inst. Henri Poincar 13, 99-125.

[12] Ivan Nourdin, Introduction to parisian fractional days.

[13] Ivan Nourdin, Generalized stochastic calculus and applications to fractional Brownin motion, non-parametric estimation of volatility.

[14] I. Nourdin and T. Simon (2006): correcting Newton-Cotes integrals by levy areas.

[15] L. Coutin (2006), An introction to (stochastic) calculs with respect to fractional Brownian motion.

[16] Mihai Gradinaru, Applications of stochastic calculus to the study of certain processes.

[17] Nourdin, I (2004) Discrete approximation of differential equations driven by an Holder continuons function.

[18] Young L. C (1936): An inequality of the Holder type connected with Stieltjes integration. Acta Math 67, 251, 282.