

The Orthogonality of Two-scale Three-dimensional Eight-direction Wavelet

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Abstract: The construction of wavelets is a key problem in wavelet analysis. In the background of the one-dimensional double wavelet theory and the one dimensional biorthogonal bidirectional wavelet construction theory, this paper extends the one-dimensional bidirectional wavelet to the two-scale three-dimensional eight-direction biorthogonal wavelet. By using the method of tensor products to construct higher dimensional wavelets, the two-scale three-dimensional eight-direction multi-resolution analysis, two-scale three-dimensional eight-direction scale function and wavelet function are obtained. In addition, the conditions satisfied of the orthogonal and biorthogonal properties of the two-scale three-dimensional eight-direction wavelet are studied.

Keywords: Two-scale Three-dimensional Eight-direction Wavelet,
Two-scale Three-dimensional Eight-direction Multiresolution Analysis, Orthogonal Wavelet

1. Introduction

The core problem of wavelet theory is to study of the structure and properties of wavelet. In order to make up for the deficiency of single wavelet, people put forward multi-wavelet theory, such as Gabor wavelet [1]. L. Shen et al proposed the symmetry-antisymmetric orthonormal multiwavelets [2], J Lebrun and M Vetterli studied the regularity problem of multiwavelets [3]. Yang Shouzhi et al gave the method of the constructing compactly supported orthonormal multiscale functions [4-8], and J Lian et al studied the orthogonal multiwavelet problem [9-12]. Professor Yang proposed the concepts of bidirectional subdivision equation and bidirectional wavelet, and discussed the orthogonality, approximation order and regularity of bidirectional subdivision function [13-14]. Especially, the bidirectional addition function is the extension of

the single scale function. Based on the construction algorithm of a-scale three-dimensional eight-direction plus fine wave scale function [15] and biorthogonal a-scale three-dimensional eight-direction scale function and wavelet function [16], this paper extended the one-dimensional bidirectional wavelet to the two-scale three-dimensional eight-direction biorthogonal wavelet by tensor product. Furthermore, the two-scale three-dimensional eight-direction multi-resolution analysis is given, the construction of two-scale three-dimensional eight-direction scale function and wavelet function is studied, and the two-scale three-dimensional eight-direction wavelet under orthogonal and biorthogonal conditions is obtained.

Introduction mark: $\forall g(x_1, x_2, x_3), f(x_1, x_2, x_3) \in L^2(R^3)$, and define the inner product as follows:

$$\langle g(x_1, x_2, x_3), f(x_1, x_2, x_3) \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x_1, x_2, x_3) \overline{f(x_1, x_2, x_3)} dx_1 dx_2 dx_3 \quad (1)$$

Where $\overline{f(x_1, x_2, x_3)}$ is the complex conjugate of $f(x_1, x_2, x_3)$. The Fourier transform of $g(x_1, x_2, x_3)$ is defined as:

$$\hat{g}(\omega_1, \omega_2, \omega_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i(x_1\omega_1 + x_2\omega_2 + x_3\omega_3)} g(x_1, x_2, x_3) dx_1 dx_2 dx_3 \quad (2)$$

The tensor product is defined as follows: let G, F and J are the three unary function spaces, the basis of G is $\{g_k(x)\}_{k \in \mathbb{Z}}$, the basis of F is $\{f_k(y)\}_{k \in \mathbb{Z}}$, and the basis of J is $\{j_k(z)\}_{k \in \mathbb{Z}}$, therefore, the tensor product space of G, F and J is a ternary function space with $\{g_k(x)f_k(y)j_k(z)\}_{k \in \mathbb{Z}}$ as the basis, namely, $g \otimes f \otimes j$; Thus: $\varphi(x, y, z) = \varphi(x)\varphi(y)\varphi(z)$ is true for a function of three variables.

2. Two-Scale Three-dimensional Eight-direction Multi-resolution Analysis

Definition 2.1. Let the scale function of two-scale three-dimensional eight-direction wavelet be set $\varphi(x_1, x_2, x_3) \in L^2(\mathbb{R}^3)$, define the subspace sequence $\{V_j\}_{j \in \mathbb{Z}} \subset L^2(\mathbb{R}^3)$:

$$V_j = \text{Clos}_{L^2(\mathbb{R}^3)} \langle 2^j \varphi(2^j x_1 - k_1, 2^j x_2 - k_2, 2^j x_3 - k_3), \\ 2^j \varphi(k_1 - 2^j x_1, 2^j x_2 - k_2, 2^j x_3 - k_3), \\ 2^j \varphi(2^j x_1 - k_1, k_2 - 2^j x_2, 2^j x_3 - k_3), \\ 2^j \varphi(2^j x_1 - k_1, 2^j x_2 - k_2, k_3 - 2^j x_3), \\ 2^j \varphi(k_1 - 2^j x_1, k_2 - 2^j x_2, 2^j x_3 - k_3), \\ 2^j \varphi(k_1 - 2^j x_1, 2^j x_2 - k_2, k_3 - 2^j x_3), \\ 2^j \varphi(2^j x_1 - k_1, k_2 - 2^j x_2, k_3 - 2^j x_3), \\ 2^j \varphi(k_1 - 2^j x_1, k_2 - 2^j x_2, k_3 - 2^j x_3) : k_1, k_2, k_3 \in \mathbb{Z} \rangle$$

(3)

Then a multi-resolution analysis in $L^2(\mathbb{R}^3)$ is generated with $(MRA)\{V_j\}_{j \in \mathbb{Z}}$ has and only has $\{V_j\}_{j \in \mathbb{Z}}$, defined by Equation (3) satisfies:

$$(i) \cdots \subset V_{-1} \subset V_0 \subset V_2 \subset \cdots;$$

$$(ii) \text{Clos}_{L^2(\mathbb{R}^3)} \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}^3);$$

$$(iii) \bigcap_{j \in \mathbb{Z}} V_j = \{0\};$$

$$(iv) f(x_1, x_2, x_3) \in V_j \Leftrightarrow f(2x_1, 2x_2, 2x_3) \in V_{j+1};$$

(V) There exists a scale function $\varphi(x_1, x_2, x_3)$ in $L^2(\mathbb{R}^3)$, make the set of functions

$$\{\varphi(x_1 - k_1, x_2 - k_2, x_3 - k_3), \\ \varphi(k_1 - x_1, x_2 - k_2, x_3 - k_3), \\ \varphi(x_1 - k_1, k_2 - x_2, x_3 - k_3), \\ \varphi(x_1 - k_1, x_2 - k_2, k_3 - x_3), \\ \varphi(k_1 - x_1, k_2 - x_2, x_3 - k_3), \\ \varphi(k_1 - x_1, x_2 - k_2, k_3 - x_3), \\ \varphi(x_1 - k_1, k_2 - x_2, k_3 - x_3), \\ \varphi(k_1 - x_1, k_2 - x_2, k_3 - x_3) : k_1, k_2, k_3 \in \mathbb{Z}\}$$

It's a Riesz basis of V_0 . Hence, two constants $0 < C_1 \leq C_2 < \infty$, can be found:

$$C_1 \left\| \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3} \overline{c_{k_1, k_2, k_3}} \right\|_2^2 \leq \left\| \begin{aligned} & \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{+,+,+} \varphi(x_1 - k_1, x_2 - k_2, x_3 - k_3) \\ & + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{+,-,-} \varphi(x_1 - k_1, x_2 - k_2, k_3 - x_3) \\ & + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{+,-,+} \varphi(x_1 - k_1, k_2 - x_2, x_3 - k_3) \\ & + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{+,-,-} \varphi(x_1 - k_1, k_2 - x_2, k_3 - x_3) \\ & + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{-,+,+} \varphi(k_1 - x_1, x_2 - k_2, x_3 - k_3) \\ & + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{-,+,+} \varphi(k_1 - x_1, x_2 - k_2, k_3 - x_3) \\ & + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{-,-,+} \varphi(k_1 - x_1, k_2 - x_2, x_3 - k_3) \\ & + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3}^{-,-,-} \varphi(k_1 - x_1, k_2 - x_2, k_3 - x_3) \end{aligned} \right\|_2^2 \leq C_2 \left\| \sum_{k_1, k_2, k_3 \in \mathbb{Z}} c_{k_1, k_2, k_3} \overline{c_{k_1, k_2, k_3}} \right\|_2^2 \quad (4)$$

Where

$$\{c_{k_1, k_2, k_3}\} = \left\{ \begin{aligned} & c_{k_1, k_2, k_3}^{+,+,+}, c_{k_1, k_2, k_3}^{+,-,-}, c_{k_1, k_2, k_3}^{+,-,+}, c_{k_1, k_2, k_3}^{+,-,-} \\ & c_{k_1, k_2, k_3}^{-,+,+}, c_{k_1, k_2, k_3}^{-,+,+}, c_{k_1, k_2, k_3}^{-,-,+}, c_{k_1, k_2, k_3}^{-,-,-} \end{aligned} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}$$

is an any sequence in $l^2(\mathbb{Z}^3)$, and equation (4) can be called a stable condition.

Definition 2.2. The tensor product of three unitary bidirectional simple wavelets $\varphi(x), \varphi(y), \varphi(z)$ is used to construct three-dimensional eight-direction bidirectional

wavelets in three-dimensional space.

Let the bidirectional simple wavelet subdivision function $\varphi(x), \varphi(y), \varphi(z)$ satisfying the subdivision equation as follows:

$$\varphi(x) = \sum_{k \in \mathbb{Z}} p_{1,k}^+ \varphi(2x - k) + \sum_{k \in \mathbb{Z}} p_{1,k}^- \varphi(k - 2x) \quad (5)$$

$$\varphi(y) = \sum_{l \in \mathbb{Z}} p_{2,l}^+ \varphi(2y - l) + \sum_{l \in \mathbb{Z}} p_{2,l}^- \varphi(l - 2y) \quad (6)$$

$$\varphi(z) = \sum_{t \in \mathbb{Z}} p_{3,t}^+ \varphi(2z - t) + \sum_{t \in \mathbb{Z}} p_{3,t}^- \varphi(t - 2z) \quad (7)$$

It can be obtained from $\varphi(x, y, z) = \varphi(x)\varphi(y)\varphi(z)$:

$$\begin{aligned} \varphi(x, y, z) &= \left\{ \sum_{k \in \mathbb{Z}} p_{1,k}^+ \varphi(2x - k) + \sum_{k \in \mathbb{Z}} p_{1,k}^- \varphi(k - 2x) \right\} \otimes \left\{ \sum_{l \in \mathbb{Z}} p_{2,l}^+ \varphi(2y - l) + \sum_{l \in \mathbb{Z}} p_{2,l}^- \varphi(l - 2y) \right\} \otimes \left\{ \sum_{t \in \mathbb{Z}} p_{3,t}^+ \varphi(2z - t) + \sum_{t \in \mathbb{Z}} p_{3,t}^- \varphi(t - 2z) \right\} \\ &= \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^+ p_{2,l}^+ p_{3,t}^+ \varphi(2x - k, 2y - l, 2z - t) + \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^+ p_{2,l}^+ p_{3,t}^- \varphi(2x - k, 2y - l, t - 2z) \\ &\quad + \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^+ p_{2,l}^- p_{3,t}^+ \varphi(2x - k, l - 2y, 2z - t) + \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^+ p_{2,l}^- p_{3,t}^- \varphi(2x - k, l - 2y, t - 2z) \\ &\quad + \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^- p_{2,l}^+ p_{3,t}^+ \varphi(k - 2x, 2y - l, 2z - t) + \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^- p_{2,l}^+ p_{3,t}^- \varphi(k - 2x, 2y - l, t - 2z) \\ &\quad + \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^- p_{2,l}^- p_{3,t}^+ \varphi(k - 2x, l - 2y, 2z - t) + \sum_{k,l,t \in \mathbb{Z}} p_{1,k}^- p_{2,l}^- p_{3,t}^- \varphi(k - 2x, l - 2y, t - 2z) \end{aligned} \quad (8)$$

Select the corresponding sequence from above to get:

$$\begin{aligned} &\left\{ p_{k_1, k_2, k_3}^{+,+,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ p_{k_1, k_2, k_3}^{+,+,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ p_{k_1, k_2, k_3}^{+,-,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ p_{k_1, k_2, k_3}^{+,-,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \\ &\left\{ p_{k_1, k_2, k_3}^{-,+,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ p_{k_1, k_2, k_3}^{-,+,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ p_{k_1, k_2, k_3}^{-,-,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ p_{k_1, k_2, k_3}^{-,-,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}} \end{aligned}$$

Then we get:

$$\begin{aligned} \varphi(x_1, x_2, x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,+,+} \varphi(2x_1 - k_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,+,-} \varphi(2x_1 - k_1, 2x_2 - k_2, k_3 - 2x_3) \\ &\quad + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,-,+} \varphi(2x_1 - k_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,-,-} \varphi(2x_1 - k_1, k_2 - 2x_2, k_3 - 2x_3) \\ &\quad + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,+,+} \varphi(k_1 - 2x_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,+,-} \varphi(k_1 - 2x_1, 2x_2 - k_2, k_3 - 2x_3) \\ &\quad + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,-,+} \varphi(k_1 - 2x_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,-,-} \varphi(k_1 - 2x_1, k_2 - 2x_2, k_3 - 2x_3) \end{aligned} \quad (9)$$

If equation (9) has a good solution, then the result of three-dimensional wavelet is an example of a two-scale three-dimensional eight-direction subdivision function.

If equation (9) can be reduced to a three-dimensional subdivision function, it is required that $p_{k_1, k_2, k_3}^{+,+,+}$, $p_{k_1, k_2, k_3}^{+,+,-}$, $p_{k_1, k_2, k_3}^{+,-,+}$, $p_{k_1, k_2, k_3}^{+,-,-}$, $p_{k_1, k_2, k_3}^{-,+,+}$, $p_{k_1, k_2, k_3}^{-,+,-}$, $p_{k_1, k_2, k_3}^{-,-,+}$, $p_{k_1, k_2, k_3}^{-,-,-}$ are 0. Let

the corresponding three-dimensional eight-direction two-scale function in $\{V_j\}_{j \in \mathbb{Z}}$ is $\varphi(x_1, x_2, x_3)$, the two-scale function of the corresponding tight support is $\tilde{\varphi}(x_1, x_2, x_3)$, then:

$$\begin{aligned} \tilde{\varphi}(x_1, x_2, x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,+,+} \tilde{\varphi}(2x_1 - k_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,+,-} \tilde{\varphi}(2x_1 - k_1, 2x_2 - k_2, k_3 - 2x_3) \\ &\quad + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,-,+} \tilde{\varphi}(2x_1 - k_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,-,-} \tilde{\varphi}(2x_1 - k_1, k_2 - 2x_2, k_3 - 2x_3) \\ &\quad + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,+,+} \tilde{\varphi}(k_1 - 2x_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,+,-} \tilde{\varphi}(k_1 - 2x_1, 2x_2 - k_2, k_3 - 2x_3) \\ &\quad + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,-,+} \tilde{\varphi}(k_1 - 2x_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,-,-} \tilde{\varphi}(k_1 - 2x_1, k_2 - 2x_2, k_3 - 2x_3) \end{aligned} \quad (10)$$

Then it can be called the sequence: $\{p_{k_1, k_2, k_3}^{+,+,+}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ and $\{p_{k_1, k_2, k_3}^{-,-,+}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ are negative, $\{\tilde{p}_{k_1, k_2, k_3}^{+,+,+}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ are three positive mask symbols; $\{p_{k_1, k_2, k_3}^{+,-,+}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ and $\{\tilde{p}_{k_1, k_2, k_3}^{+,-,+}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ are positive, positive and negative mask symbols; $\{p_{k_1, k_2, k_3}^{+,-,-}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ and $\{\tilde{p}_{k_1, k_2, k_3}^{+,-,-}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ are positive, negative and positive mask symbols; $\{p_{k_1, k_2, k_3}^{-,+,-}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ and $\{\tilde{p}_{k_1, k_2, k_3}^{-,+,-}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ are positive, negative and negative mask symbols; $\{p_{k_1, k_2, k_3}^{-,-,+}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ and $\{\tilde{p}_{k_1, k_2, k_3}^{-,-,+}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ are negative, positive, negative mask symbols; $\{p_{k_1, k_2, k_3}^{-,-,-}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ and $\{\tilde{p}_{k_1, k_2, k_3}^{-,-,-}\}_{k_1, k_2, k_3 \in \mathbb{Z}}$ are three negative mask symbols. Fourier transform of equations (9) and (10) can be obtained:

$$\left\{ \begin{aligned} \hat{\phi}(\omega_1, \omega_2, \omega_3) &= p^{+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + p^{+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\ &\quad + p^{+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + p^{+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\ &\quad + p^{-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + p^{-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\ &\quad + p^{-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + p^{-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\ \hat{\tilde{\phi}}(\omega_1, \omega_2, \omega_3) &= \hat{p}^{+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{p}^{+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\ &\quad + \hat{p}^{+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{p}^{+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\ &\quad + \hat{p}^{-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{p}^{-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\ &\quad + \hat{p}^{-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{p}^{-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) \end{aligned} \right. \quad (11)$$

There in:

$$\begin{aligned} p^{+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,+,+} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; p^{+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) = \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,-,+} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\ p^{+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,-,+} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; p^{+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) = \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,-,-} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\ p^{-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,+,-} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; p^{-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) = \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,+,-} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\ p^{-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,-,+} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; p^{-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) = \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,-,-} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \end{aligned}$$

The deformation of equations (9) and (10) is as follows:

$$\left. \begin{aligned} \varphi(x_1, -x_2, x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+, +, +} \varphi(2x_1 - k_1, -2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+, +, -} \varphi(2x_1 - k_1, -2x_2 - k_2, k_3 - 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+, -, +} \varphi(2x_1 - k_1, k_2 + 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+, -, -} \varphi(2x_1 - k_1, k_2 + 2x_2, k_3 - 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-, +, +} \varphi(k_1 - 2x_1, -2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-, +, -} \varphi(k_1 - 2x_1, -2x_2 - k_2, k_3 - 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-, -, +} \varphi(k_1 - 2x_1, k_2 + 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-, -, -} \varphi(k_1 - 2x_1, k_2 + 2x_2, k_3 - 2x_3) \\ \tilde{\varphi}(x_1, -x_2, x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+, +, +} \tilde{\varphi}(2x_1 - k_1, -2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+, +, -} \tilde{\varphi}(2x_1 - k_1, -2x_2 - k_2, k_3 - 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+, -, +} \tilde{\varphi}(2x_1 - k_1, k_2 + 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+, -, -} \tilde{\varphi}(2x_1 - k_1, k_2 + 2x_2, k_3 - 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-, +, +} \tilde{\varphi}(k_1 - 2x_1, -2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-, +, -} \tilde{\varphi}(k_1 - 2x_1, -2x_2 - k_2, k_3 - 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-, -, +} \tilde{\varphi}(k_1 - 2x_1, k_2 + 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-, -, -} \tilde{\varphi}(k_1 - 2x_1, k_2 + 2x_2, k_3 - 2x_3) \end{aligned} \right\} \quad (13)$$

$$\left\{ \begin{aligned} \varphi(x_1, x_2, -x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,+,+} \varphi(2x_1 - k_1, 2x_2 - k_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,+,-} \varphi(2x_1 - k_1, 2x_2 - k_2, k_3 + 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,-,+} \varphi(2x_1 - k_1, k_2 - 2x_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{+,-,-} \varphi(2x_1 - k_1, k_2 - 2x_2, k_3 + 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,+,+} \varphi(k_1 - 2x_1, 2x_2 - k_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,+,-} \varphi(k_1 - 2x_1, 2x_2 - k_2, k_3 + 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,-,+} \varphi(k_1 - 2x_1, k_2 - 2x_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} p_{k_1, k_2, k_3}^{-,-,-} \varphi(k_1 - 2x_1, k_2 - 2x_2, k_3 + 2x_3) \\ \tilde{\varphi}(x_1, x_2, -x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,+,+} \tilde{\varphi}(2x_1 - k_1, 2x_2 - k_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,+,-} \tilde{\varphi}(2x_1 - k_1, 2x_2 - k_2, k_3 + 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,-,+} \tilde{\varphi}(2x_1 - k_1, k_2 - 2x_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{+,-,-} \tilde{\varphi}(2x_1 - k_1, k_2 - 2x_2, k_3 + 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,+,+} \tilde{\varphi}(k_1 - 2x_1, 2x_2 - k_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,+,-} \tilde{\varphi}(k_1 - 2x_1, 2x_2 - k_2, k_3 + 2x_3) \\ &+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,-,+} \tilde{\varphi}(k_1 - 2x_1, k_2 - 2x_2, -2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{p}_{k_1, k_2, k_3}^{-,-,-} \tilde{\varphi}(k_1 - 2x_1, k_2 - 2x_2, k_3 + 2x_3) \end{aligned} \right. \quad (14)$$

$$\left\{ \begin{aligned} \hat{\phi}(\omega_1, -\omega_2, -\omega_3) &= p^{+,+,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{+,+,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + p^{+,-,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{+,-,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + p^{-,+,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{-,+,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + p^{-,-,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{-,-,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \end{aligned} \right. \quad (24)$$

$$\left\{ \begin{aligned} \hat{\hat{\phi}}(\omega_1, -\omega_2, -\omega_3) &= \hat{p}^{+,+,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{+,+,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + \hat{p}^{+,-,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{+,-,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + \hat{p}^{-,+,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{-,+,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + \hat{p}^{-,-,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{-,-,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \end{aligned} \right.$$

$$\left\{ \begin{aligned} \hat{\phi}(-\omega_1, -\omega_2, -\omega_3) &= p^{+,+,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{+,+,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + p^{+,-,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{+,-,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + p^{-,+,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{-,+,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + p^{-,-,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + p^{-,-,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \end{aligned} \right. \quad (25)$$

$$\left\{ \begin{aligned} \hat{\hat{\phi}}(-\omega_1, -\omega_2, -\omega_3) &= \hat{p}^{+,+,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{+,+,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + \hat{p}^{+,-,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{+,-,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + \hat{p}^{-,+,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{-,+,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ &\quad + \hat{p}^{-,-,+}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) + \hat{p}^{-,-,-}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})\hat{\hat{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \end{aligned} \right.$$

If the:

$$\begin{aligned} \phi(x_1, x_2, x_3) &= [\phi(x_1, x_2, x_3), \phi(-x_1, x_2, x_3), \phi(x_1, -x_2, x_3), \phi(x_1, x_2, -x_3), \\ &\quad \phi(-x_1, -x_2, x_3), \phi(-x_1, x_2, -x_3), \phi(x_1, -x_2, -x_3), \phi(-x_1, -x_2, -x_3)]^T \end{aligned} \quad (26)$$

$$\begin{aligned} \tilde{\phi}(x_1, x_2, x_3) &= [\tilde{\phi}(x_1, x_2, x_3), \tilde{\phi}(-x_1, x_2, x_3), \tilde{\phi}(x_1, -x_2, x_3), \tilde{\phi}(x_1, x_2, -x_3), \\ &\quad \tilde{\phi}(-x_1, -x_2, x_3), \tilde{\phi}(-x_1, x_2, -x_3), \tilde{\phi}(x_1, -x_2, -x_3), \tilde{\phi}(-x_1, -x_2, -x_3)]^T \end{aligned} \quad (27)$$

Where 'T' represents the transpose of the matrix, and there can be partitioned matrices:

$$A = \begin{pmatrix} p^{+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{+,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ p^{+,+,-}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{+,+,+}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{-,-,+}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{-,+,-}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ p^{+,-,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{-,-,-}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{+,+,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) & p^{-,-,+}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ p^{+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) & p^{-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) & p^{-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) & p^{+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \end{pmatrix}$$

$$\begin{aligned}
B &= \begin{pmatrix} p^{-,+,\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{-,+,-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{-,-,\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{-,-,-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}} \\ p^{+,-,\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{+,-,-\frac{\omega_1}{2},\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{+,-,\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{+,-,-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} \\ p^{+,+,\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{+,+,-\frac{\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{+,-,\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} \\ p^{-,-,\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} \end{pmatrix} \\
C &= \begin{pmatrix} p^{-,+,\frac{-\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{-,+,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{-,+,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} \\ p^{-,+,\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,+,\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,+,-\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,+,-\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} \\ p^{-,-,\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} \\ p^{-,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{-,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} \end{pmatrix} \\
D &= \begin{pmatrix} p^{+,+,\frac{-\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{+,+,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} & p^{+,+,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},\frac{\omega_3}{2}} \\ p^{+,-,\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{-\omega_1}{2},\frac{\omega_2}{2},-\frac{\omega_3}{2}} \\ p^{+,-,\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} \\ p^{+,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} & p^{+,-,-\frac{-\omega_1}{2},-\frac{\omega_2}{2},-\frac{\omega_3}{2}} \end{pmatrix}
\end{aligned}$$

Thus, the symbols of the enhanced mask in Equations (26) and (27) can be obtained.

3. Biorthogonal of the Two-scale Three-dimensional Eight-direction Scale Functions and Wavelet Functions

Definition 3.1 If the three-dimensional eight-direction scale function $\varphi(x_1, x_2, x_3)$ [16] is orthogonal, then the following conditions are true :

$$\begin{aligned}
&\langle \varphi(x_1, x_2, x_3), \varphi(x_1 - k_1, x_2 - k_2, x_3 - k_3) \rangle = \delta_{0,k_1} \delta_{0,k_2} \delta_{0,k_3}; \\
&\langle \varphi(x_1, x_2, x_3), \varphi(k_1 - x_1, x_2 - k_2, x_3 - k_3) \rangle = 0; \\
&\langle \varphi(x_1, x_2, x_3), \varphi(x_1 - k_1, k_2 - x_2, x_3 - k_3) \rangle = 0; \\
&\langle \varphi(x_1, x_2, x_3), \varphi(x_1 - k_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
&\langle \varphi(x_1, x_2, x_3), \varphi(k_1 - x_1, k_2 - x_2, x_3 - k_3) \rangle = 0; \\
&\langle \varphi(x_1, x_2, x_3), \varphi(k_1 - x_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
&\langle \varphi(x_1, x_2, x_3), \varphi(x_1 - k_1, k_2 - x_2, k_3 - x_3) \rangle = 0; \\
&\langle \varphi(x_1, x_2, x_3), \varphi(k_1 - x_1, k_2 - x_2, k_3 - x_3) \rangle = 0;
\end{aligned} \tag{28}$$

Definition 3.2 If the three-dimensional eight-direction scale functions $\varphi(x_1, x_2, x_3)$ and $\tilde{\varphi}(x_1, x_2, x_3)$ [16] are biorthogonal, then the following equations are satisfied :

$$\begin{aligned}
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(x_1 - k_1, x_2 - k_2, x_3 - k_3) \rangle &= \delta_{0, k_1} \delta_{0, k_2} \delta_{0, k_3}; \\
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(k_1 - x_1, x_2 - k_2, x_3 - k_3) \rangle &= 0; \\
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(x_1 - k_1, k_2 - x_2, x_3 - k_3) \rangle &= 0; \\
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(x_1 - k_1, x_2 - k_2, k_3 - x_3) \rangle &= 0; \\
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(k_1 - x_1, k_2 - x_2, x_3 - k_3) \rangle &= 0; \\
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(k_1 - x_1, x_2 - k_2, k_3 - x_3) \rangle &= 0; \\
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(x_1 - k_1, k_2 - x_2, k_3 - x_3) \rangle &= 0; \\
\langle \varphi(x_1, x_2, x_3), \tilde{\varphi}(k_1 - x_1, k_2 - x_2, k_3 - x_3) \rangle &= 0;
\end{aligned} \tag{29}$$

Definition 3.3 If $\{V_j\}_{j \in \mathbb{Z}}$ is a three-dimensional eight-direction multi-resolution analysis of $\varphi(x_1, x_2, x_3)$ scale function and $\{W_j\}_{j \in \mathbb{Z}}$ is the orthogonal complement space of V_j in V_{j+1} as well as $L^2(R^3) = \bigoplus_{j \in \mathbb{Z}} W_j$, where $W_j = W_j^{(1)} \oplus W_j^{(2)} \oplus W_j^{(3)} \oplus W_j^{(4)} \oplus W_j^{(5)} \oplus W_j^{(6)} \oplus W_j^{(7)}$, $W_j^{(\gamma)} (\gamma = 1, 2, 3, 4, 5, 6, 7)$ is orthogonal to each other. Thus, there exists a ternary compact supported wavelet function $\psi^\gamma(x_1, x_2, x_3) : \gamma = 1, 2, 3, 4, 5, 6, 7$, making the function set:

$$\begin{aligned}
&\{\psi^\gamma(x_1 - k_1, x_2 - k_2, x_3 - k_3), \psi^\gamma(k_1 - x_1, x_2 - k_2, x_3 - k_3), \\
&\psi^\gamma(x_1 - k_1, k_2 - x_2, x_3 - k_3), \psi^\gamma(x_1 - k_1, x_2 - k_2, k_3 - x_3), \\
&\psi^\gamma(k_1 - x_1, k_2 - x_2, x_3 - k_3), \psi^\gamma(k_1 - x_1, x_2 - k_2, k_3 - x_3), \\
&\psi^\gamma(x_1 - k_1, k_2 - x_2, k_3 - x_3), \psi^\gamma(k_1 - x_1, k_2 - x_2, k_3 - x_3) \\
&: \gamma = 1, 2, 3, 4, 5, 6, 7; k_1, k_2, k_3 \in \mathbb{Z}\}
\end{aligned}$$

It forms an orthonormal basis for W_0 . The tightly supported wavelet function $\tilde{\psi}^\gamma(x_1, x_2, x_3) : \gamma = 1, 2, 3, 4, 5, 6, 7$, corresponding to $\psi^\gamma(x_1, x_2, x_3) : \gamma = 1, 2, 3, 4, 5, 6, 7$, satisfies:

$$\begin{aligned}
\langle \varphi(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, x_2 - k_2, x_3 - k_3) \rangle &= \langle \varphi(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, x_2 - k_2, x_3 - k_3) \rangle = 0; \\
\langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, x_2 - k_2, x_3 - k_3) \rangle &= \langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, x_2 - k_2, x_3 - k_3) \rangle = 0; \\
\langle \varphi(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, k_2 - x_2, x_3 - k_3) \rangle &= \langle \varphi(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
\langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, k_2 - x_2, x_3 - k_3) \rangle &= \langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
\langle \varphi(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, k_2 - x_2, x_3 - k_3) \rangle &= \langle \varphi(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
\langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, k_2 - x_2, x_3 - k_3) \rangle &= \langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
\langle \varphi(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, k_2 - x_2, k_3 - x_3) \rangle &= \langle \varphi(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, k_2 - x_2, k_3 - x_3) \rangle = 0; \\
\langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(x_1 - k_1, k_2 - x_2, k_3 - x_3) \rangle &= \langle \tilde{\varphi}(x_1, x_2, x_3), \psi^\gamma(k_1 - x_1, k_2 - x_2, k_3 - x_3) \rangle = 0;
\end{aligned}$$

$$\begin{aligned}
&\langle \psi^k(x_1, x_2, x_3), \psi^t(x_1 - k_1, x_2 - k_2, x_3 - k_3) \rangle = \delta_{k,t} \delta_{0,k_1} \delta_{0,k_2} \delta_{0,k_3}; \langle \psi^k(x_1, x_2, x_3), \psi^t(k_1 - x_1, x_2 - k_2, x_3 - k_3) \rangle = 0; \\
&\langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(x_1 - k_1, x_2 - k_2, x_3 - k_3) \rangle = \delta_{k,t} \delta_{0,k_1} \delta_{0,k_2} \delta_{0,k_3}; \langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(k_1 - x_1, x_2 - k_2, x_3 - k_3) \rangle = 0; \\
&\langle \psi^k(x_1, x_2, x_3), \psi^t(x_1 - k_1, k_2 - x_2, x_3 - k_3) \rangle = 0; \langle \psi^k(x_1, x_2, x_3), \psi^t(x_1 - k_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
&\langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(x_1 - k_1, k_2 - x_2, x_3 - k_3) \rangle = 0; \langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(x_1 - k_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
&\langle \psi^k(x_1, x_2, x_3), \psi^t(k_1 - x_1, k_2 - x_2, x_3 - k_3) \rangle = 0; \langle \psi^k(x_1, x_2, x_3), \psi^t(k_1 - x_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
&\langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(k_1 - x_1, k_2 - x_2, x_3 - k_3) \rangle = 0; \langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(k_1 - x_1, x_2 - k_2, k_3 - x_3) \rangle = 0; \\
&\langle \psi^k(x_1, x_2, x_3), \psi^t(x_1 - k_1, k_2 - x_2, k_3 - x_3) \rangle = 0; \langle \psi^k(x_1, x_2, x_3), \psi^t(k_1 - x_1, k_2 - x_2, k_3 - x_3) \rangle = 0; \\
&\langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(x_1 - k_1, k_2 - x_2, k_3 - x_3) \rangle = 0; \langle \tilde{\psi}^k(x_1, x_2, x_3), \tilde{\psi}^t(k_1 - x_1, k_2 - x_2, k_3 - x_3) \rangle = 0;
\end{aligned} \tag{30}$$

Among them: $\gamma, k, t = 1, 2, 3, 4, 5, 6, 7$.

Hence there is a sequence:

$$\begin{aligned}
&\left(\left\{ q_{k_1, k_2, k_3}^{\gamma+,+,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ q_{k_1, k_2, k_3}^{\gamma+,+,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \right. \\
&\left. \left\{ q_{k_1, k_2, k_3}^{\gamma+,-,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ q_{k_1, k_2, k_3}^{\gamma+,-,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \right. \\
&\left. \left\{ q_{k_1, k_2, k_3}^{\gamma-,+,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ q_{k_1, k_2, k_3}^{\gamma-,+,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \right. \\
&\left. \left\{ q_{k_1, k_2, k_3}^{\gamma-,-,+} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}}, \left\{ q_{k_1, k_2, k_3}^{\gamma-,-,-} \right\}_{k_1, k_2, k_3 \in \mathbb{Z}} \right) \in l^2(\mathbb{Z}^3)
\end{aligned}$$

To satisfy:

$$\begin{aligned}
\psi^\gamma(x_1, x_2, x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma+,+,+} \varphi(2x_1 - k_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma+,+,-} \varphi(2x_1 - k_1, 2x_2 - k_2, k_3 - 2x_3) \\
&+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma+,-,+} \varphi(2x_1 - k_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma+,-,-} \varphi(2x_1 - k_1, k_2 - 2x_2, k_3 - 2x_3) \\
&+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma-,+,+} \varphi(k_1 - 2x_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma-,+,-} \varphi(k_1 - 2x_1, 2x_2 - k_2, k_3 - 2x_3) \\
&+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma-,-,+} \varphi(k_1 - 2x_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q_{k_1, k_2, k_3}^{\gamma-,-,-} \varphi(k_1 - 2x_1, k_2 - 2x_2, k_3 - 2x_3)
\end{aligned} \tag{31}$$

The same is true for $\tilde{\psi}^\gamma(x_1, x_2, x_3) : \gamma = 1, 2, 3, 4, 5, 6, 7$.

$$\begin{aligned}
\tilde{\psi}^\gamma(x_1, x_2, x_3) &= \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma+,+,+} \tilde{\varphi}(2x_1 - k_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma+,+,-} \tilde{\varphi}(2x_1 - k_1, 2x_2 - k_2, k_3 - 2x_3) \\
&+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma+,-,+} \tilde{\varphi}(2x_1 - k_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma+,-,-} \tilde{\varphi}(2x_1 - k_1, k_2 - 2x_2, k_3 - 2x_3) \\
&+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma-,+,+} \tilde{\varphi}(k_1 - 2x_1, 2x_2 - k_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma-,+,-} \tilde{\varphi}(k_1 - 2x_1, 2x_2 - k_2, k_3 - 2x_3) \\
&+ \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma-,-,+} \tilde{\varphi}(k_1 - 2x_1, k_2 - 2x_2, 2x_3 - k_3) + \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}_{k_1, k_2, k_3}^{\gamma-,-,-} \tilde{\varphi}(k_1 - 2x_1, k_2 - 2x_2, k_3 - 2x_3)
\end{aligned} \tag{32}$$

The Fourier transform of (31) and (32) can be obtained:

$$\left\{ \begin{aligned}
\hat{\psi}^{\gamma}(\omega_1, \omega_2, \omega_3) &= Q^{\gamma+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + Q^{\gamma+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\
&\quad + Q^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + Q^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\
&\quad + Q^{\gamma-,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\
&\quad + Q^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + Q^{\gamma-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\phi}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\
\hat{\tilde{\psi}}^{\gamma}(\omega_1, \omega_2, \omega_3) &= \hat{Q}^{\gamma+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{Q}^{\gamma+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\
&\quad + \hat{Q}^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{Q}^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\
&\quad + \hat{Q}^{\gamma-,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{Q}^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, \frac{\omega_2}{2}, -\frac{\omega_3}{2}) \\
&\quad + \hat{Q}^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, \frac{\omega_3}{2}) + \hat{Q}^{\gamma-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \hat{\tilde{\phi}}(-\frac{\omega_1}{2}, -\frac{\omega_2}{2}, -\frac{\omega_3}{2})
\end{aligned} \right. \quad (33)$$

Among them:

$$\begin{aligned}
Q^{\gamma+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma+,+,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & Q^{\gamma+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma+,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\
Q^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma+,-,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & Q^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma+,-,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\
Q^{\gamma-,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma-,+,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma-,+,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\
Q^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma-,-,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & Q^{\gamma-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} q^{\gamma-,-,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})};
\end{aligned}$$

$\psi^{\gamma}(x_1, x_2, x_3)$ is called the three positive mask symbols; positive, positive and negative mask symbols; positive, negative and positive mask symbols; positive, negative and negative mask symbols; negative, positive and positive mask symbols; negative, positive and negative mask symbols; negative, negative and positive mask symbols; three negative mask symbols, respectively. The corresponding $\tilde{\psi}^{\gamma}(x_1, x_2, x_3) : \gamma = 1, 2, 3, 4, 5, 6, 7$ are three positive mask symbols; positive, positive and negative mask symbols; positive, negative and negative mask symbols; positive, negative and positive mask symbols; negative, positive and positive mask symbols; negative, positive and negative mask symbols; negative, negative and positive mask symbols; three negative mask symbols, respectively. As follows:

$$\begin{aligned}
\tilde{Q}^{\gamma+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma+,+,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & \tilde{Q}^{\gamma+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma+,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\
\tilde{Q}^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma+,-,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & \tilde{Q}^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma+,-,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\
\tilde{Q}^{\gamma-,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma-,+,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & \tilde{Q}^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma-,+,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; \\
\tilde{Q}^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma-,-,+}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})}; & \tilde{Q}^{\gamma-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) &= \frac{1}{8} \sum_{k_1, k_2, k_3 \in \mathbb{Z}} \tilde{q}^{\gamma-,-,-}_{k_1, k_2, k_3} e^{-i(\frac{\omega_1 k_1}{2} + \frac{\omega_2 k_2}{2} + \frac{\omega_3 k_3}{2})};
\end{aligned}$$

If the:

$$\begin{cases} \Psi^\gamma(x_1, x_2, x_3) = (\psi^\gamma(x_1, x_2, x_3), \psi^\gamma(-x_1, x_2, x_3), \psi^\gamma(x_1, -x_2, x_3), \psi^\gamma(x_1, x_2, -x_3), \\ \psi^\gamma(-x_1, -x_2, x_3), \psi^\gamma(-x_1, x_2, -x_3), \psi^\gamma(x_1, -x_2, -x_3), \psi^\gamma(-x_1, -x_2, -x_3))^T; \\ \tilde{\Psi}^\gamma(x_1, x_2, x_3) = (\tilde{\psi}^\gamma(x_1, x_2, x_3), \tilde{\psi}^\gamma(-x_1, x_2, x_3), \tilde{\psi}^\gamma(x_1, -x_2, x_3), \tilde{\psi}^\gamma(x_1, x_2, -x_3), \\ \tilde{\psi}^\gamma(-x_1, -x_2, x_3), \tilde{\psi}^\gamma(-x_1, x_2, -x_3), \tilde{\psi}^\gamma(x_1, -x_2, -x_3), \tilde{\psi}^\gamma(-x_1, -x_2, -x_3))^T; \end{cases}$$

Where 'T' is the transpose of the matrix, and there are partitioned matrix E, F, G and H whose specific form are as follows:

$$\begin{aligned} E &= \begin{pmatrix} Q^{\gamma+,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) \end{pmatrix} \\ F &= \begin{pmatrix} Q^{\gamma-,+,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,-,-}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,-,+}(\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) \end{pmatrix} \\ G &= \begin{pmatrix} Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) \\ Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) \\ Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma-,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) \end{pmatrix} \\ H &= \begin{pmatrix} Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{\omega_3}{2}) \\ Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{\omega_2}{2}, \frac{-\omega_3}{2}) \\ Q^{\gamma+,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) \\ Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) & Q^{\gamma+,+,-}(\frac{-\omega_1}{2}, \frac{-\omega_2}{2}, \frac{-\omega_3}{2}) \end{pmatrix} \end{aligned}$$

Then, the enhanced masks corresponding to $\Psi^\gamma(x_1, x_2, x_3)$ and $\tilde{\Psi}^\gamma(x_1, x_2, x_3)$ can be written as follows:

$$I^\gamma(\omega_1, \omega_2, \omega_3) = \begin{bmatrix} E & F \\ G & H \end{bmatrix}; \quad \tilde{I}^\gamma(\omega_1, \omega_2, \omega_3) = \begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{G} & \tilde{H} \end{bmatrix}$$

4. Conclusion

Based on bidirectional wavelet theory and tensor product

principle, the two-scale three-dimensional eight-direction multiresolution analysis is given. Furthermore, the construction of two-scale three-dimensional eight-direction

scale function and wavelet function is studied, and the two-scale three-dimensional eight-direction wavelets under orthogonal and biorthogonal conditions are obtained. Hereby, the one-dimension orthogonal biorthogonal wavelet is extended to two-scale three-dimension eight-dimension biorthogonal wavelet; this research may promote the theoretical study of high dimensional wavelet.

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