American Journal of Applied Mathematics
2021; 9(1): 31-37
http://www.sciencepublishinggroup.com/j/ajhc
doi: 10.11648/j.ajam.20210901.15
ISSN: 2575-7059 (Print); ISSN: 2575-5722 (Online)

A Künneth Formula for the Embedded Homology

Chong Wang[1,2]

[1]School of Mathematics and Statistics, Cangzhou Normal University, Cangzhou, China
[2]School of Mathematics, Renmin University of China, Beijing, China

Email address:
wangchong_618@163.com

To cite this article:
Chong Wang. A Künneth Formula for the Embedded Homology. *American Journal of Applied Mathematics.*
Vol. 9, No. 1, 2021, pp. 31-37. doi: 10.11648/j.ajam.20210901.15

Received: May 8, 2020; **Accepted:** April 8, 2021; **Published:** April 10, 2021

Abstract: Hypergraph is an important model for complex networks. A hypergraph can be regarded as a virtual simplicial complex with some faces missing and it is the key hub to connect the simplicial complex in topology and graph in combinatorics. The embedded homology groups of hypergraphs are new developments in mathematics in recent years, and the embedded homology groups of hypergraphs can reflect the topological and geometric characteristics of complex network which can not be reflected by the associated simplicial complex of hypergraphs. Künneth formulas describe the homology or cohomology of a product space in terms of the homology or cohomology of the factors. In this paper, we prove that the infimum chain complex of tensor products of free R-modules generated by hypergraphs is isomorphic to the tensor product of their respective infimum chain complexes, and give an analogues of Künneth formula for hypergraphs by classical algebraic Künneth formula based on the embedded homology groups of hypergraphs, which provides a theoretical basis for further study of cohomology theory of hypergraphs. In fact, the Künneth formula here can be extended to the Künneth formula of embedded homology of graded abelian groups of chain complexes, which can be used to extend the Künneth formula for digraphs with coefficients in a field.

Keywords: Hypergraphs, Embedded Homology, Associated Simplicial Complexes, Künneth Formula

1. Introduction

In topology, a hypergraph can be obtained from a simplicial complex by deleting some non-maximal simplices [2, 13]. Hypergraph is a standard mathematical network-model for many real-world data problems. For example, the coauthorship network of scientific researchers and their collaborations [12], the biological cellular networks [11], and the network of biomolecules and biomolecular interactions [14]. Hypergraph is the key hub to connect the simplicial complex in topology and graph in combinatorics, which is worth studying in theory and application [1, 2, 13].

Let $V_{\mathcal{H}}$ be a totally-ordered finite set. Let 2^V denote the powerset of V. Let $\emptyset$ denote the empty set. A *hypergraph* is a pair $(V_{\mathcal{H}}, \mathcal{H})$ where $\mathcal{H}$ is a subset of $2^V \setminus \{\emptyset\}$ [1, 13]. An element of $V_{\mathcal{H}}$ is called a *vertex* and an element of $\mathcal{H}$ is called a *hyperedge*. A hyperedge $\sigma \in \mathcal{H}$ consisting of $k+1$ vertices is called a *k-dimensional hyperedge* ($k \geq 0$), denoted as $\sigma^{(p)}$ or σ for short. Throughout this paper, we assume that each vertex in $V_{\mathcal{H}}$ appears in at least one hyperedge in $\mathcal{H}$. Hence $V_{\mathcal{H}}$ is the union $\bigcup_{\sigma\in\mathcal{H}} \sigma$, and we simply denote a hypergraph $(V_{\mathcal{H}}, \mathcal{H})$ as $\mathcal{H}$.

Let $\mathcal{H}$ be a hypergraph. The *associated simplicial complex* $\mathcal{K}_{\mathcal{H}}$ of $\mathcal{H}$ is defined as the smallest simplicial complex that $\mathcal{H}$ can be embedded in [13]. Precisely, the set of all simplices of $\mathcal{K}_{\mathcal{H}}$ consists of all the non-empty subsets $\tau \subseteq \sigma$, for all $\sigma \in \mathcal{H}$.

There are various (co)homology theories of hypergraphs. For example, A.D. Parks and S.L. Lipscomb studied the homology of the associated simplicial complex in 1991 [13]. F.R.K. Chung and R.L. Graham constructed certain cohomology for hypergraphs in a combinatorial way in 1992 [3]. E. Emtander constructed the independence simplicial complexes for hypergraphs and studied the homology of these simplicial complexes [4], and J. Johnson applied the topology of hypergraphs to study hyper-networks of complex systems in 2009 [8]. S. Bressan, J. Li, S. Ren and J. Wu defined the embedded homology of hypergraphs as well as the persistent embedded homology of sequences of hypergraphs in 2019 [2].

Let R be a principal ideal domain. Let $\mathcal{H}$ be a hypergraph. Let $R(\mathcal{H})_n$ be the finitely generated free R-module with generators of n-dimensional hyperedges in $\mathcal{H}$. Let $\mathcal{K}$ be a simplicial complex such that $\mathcal{H} \subseteq \mathcal{K}$. The *infimum chain complex* and the *supremum chain complex* of $\mathcal{H}$ are defined as

$$\mathrm{Inf}_n(R(\mathcal{H})_*) = R(\mathcal{H}_n) \cap (\partial_n)^{-1} R(\mathcal{H})_{n-1}, n \geq 0 \quad (1)$$

and

$$\mathrm{Sup}_n(R(\mathcal{H})_*) = R(\mathcal{H}_n) \cap \partial_{n+1} R(\mathcal{H})_{n+1}, n \geq 0 \quad (2)$$

respectively, where ∂_* is the boundary maps of $\mathcal{K}$ and ∂_*^{-1} denotes the pre-image of ∂_* [2]. It is proved that both $\mathrm{Inf}_n(R(\mathcal{H})_*)$ and $\mathrm{Sup}_n(R(\mathcal{H})_*)$ do not depend on the choice of the simplicial complex $\mathcal{K}$ that $\mathcal{H}$ embedded in. Therefore, $\mathcal{K}$ can be taken as the associated simplicial complex of $\mathcal{H}$. The homologies of these two chain complexes (1) and (2) are isomorphic [2, Proposition 2.4], which are defined as the *embedded homology* of $\mathcal{H}$ and denoted as $H_n(\mathrm{Inf}_*(\mathcal{H}))$ or simply $H_n(\mathcal{H})$. In particular, if the hypergraph is a simplicial complex, then the embedded homology coincides with the usual homology. Moreover, each morphism of hypergraphs from $\mathcal{H}$ to $\mathcal{H}'$ induces an homomorphism between the embedded homology $H_n(\mathcal{H}; R)$ and $H_n(\mathcal{H}'; R)$ [2, Proposition 3.7].

Künneth formulas describe the homology or cohomology of a product space in terms of the homology or cohomology of the factors. Hatcher gave the classical algebraic Künneth formula [7]. A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau studied the the Künneth formula for the path homology (with field coefficients) of digraphs [5, 6].

In this paper, we give a Künneth formula for the Embedded Homology in Theorem 4.2.

2. Preliminaries

In this section, we define the tensor product for graded abelian subgroups of chain complexes.

Firstly, we review the definition for the tensor product of chain complexes. The content of this subsection can be found in [7, Chapter 3, Section 3.B].

Let C and C' be chain complexes

$$C = \{C_n, \partial_n\}_{n\geq 0}, \quad C' = \{C'_n, \partial'_n\}_{n\geq 0}.$$

Their tensor product is a chain complex

$$C \otimes C' = \Big\{ \bigoplus_{\substack{p+q=n,\\ p,q\geq 0}} C_p \otimes C'_q, \bigoplus_{\substack{p+q=n,\\ p,q\geq 0}} \partial_p \otimes \partial'_q \Big\}_{n\geq 0}. \quad (3)$$

In (3), the tensor product of boundary maps is given by

$$(\partial_p \otimes \partial'_q)(u_p \otimes v_q) = (\partial_p u_p) \otimes v_q + (-1)^p u_p \otimes (\partial'_q v_q)$$

for any $u_p \in C_p$ and any $v_q \in C'_q$. For simplicity, we denote

$$(C \otimes C')_n = \bigoplus_{\substack{p+q=n,\\ p,q\geq 0}} C_p \otimes C'_q, \quad (\partial \otimes \partial')_n = \bigoplus_{\substack{p+q=n,\\ p,q\geq 0}} \partial_p \otimes \partial'_q.$$

Then for any $u_p \in C_p$ and any $v_q \in C'_q$,

$$(\partial \otimes \partial')_n \Big(\sum_{\substack{p+q=n,\\ p,q\geq 0}} u_p \otimes v_q \Big) = \sum_{\substack{p+q=n,\\ p,q\geq 0}} (\partial_p u_p) \otimes v_q + (-1)^p u_p \otimes (\partial'_q v_q). \quad (4)$$

It follows from (4) that

$$(\partial \otimes \partial')_n : (C \otimes C')_n \longrightarrow (C \otimes C')_{n-1}$$

is well-defined, and for any $n \geq 0$,

$$(\partial \otimes \partial')_n \circ (\partial \otimes \partial')_{n+1} = 0.$$

Hence (3) is a chain complex.

Secondly, we generalize the tensor product of chain complexes and define the tensor product for graded abelian subgroups of chain complexes.

For each $n \geq 0$, we consider abelian subgroups $D_n \subseteq C_n$ and $D'_n \subseteq C'_n$. Then we have graded abelian subgroups of the chain complexes

$$D = \{D_n\}_{n\geq 0}, \quad D' = \{D'_n\}_{n\geq 0}.$$

The tensor product of D and D' is defined as

$$D \otimes D' = \Big\{ \bigoplus_{\substack{p+q=n,\\ p,q\geq 0}} D_p \otimes D'_q \Big\}.$$

It is direct to verify that $D \otimes D'$ is a graded abelian subgroup of the chain complex $C \otimes C'$. For simplicity, we denote

$$(D \otimes D')_n = \bigoplus_{\substack{p+q=n,\\ p,q\geq 0}} D_p \otimes D'_q.$$

Lemma 2.1. For any $n \geq 0$,

$$(\partial \otimes \partial')_n (D \otimes D')_n = \sum_{\substack{p+q=n,\\ p,q\geq 0}} \partial_p D_p \otimes D'_q + D_p \otimes \partial'_q D'_q.$$

Proof. By (4), the lemma follows from a calculation

$$\begin{aligned}(\partial \otimes \partial')_n (D \otimes D')_n &= (\partial \otimes \partial')_n \Big(\bigoplus_{\substack{p+q=n,\\ p,q\geq 0}} D_p \otimes D'_q \Big) \\ &= \sum_{\substack{p+q=n,\\ p,q\geq 0}} (\partial \otimes \partial')_n (D_p \otimes D'_q) \\ &= \sum_{\substack{p+q=n,\\ p,q\geq 0}} \partial_p D_p \otimes D'_q + D_p \otimes \partial'_q D'_q.\end{aligned}$$

3. Auxiliary Results for Theorem 4.2

The finitely generated free R-module generated by all hyperedges of a hypergraph is a graded abelian group of the chain complex of its associated complex [9, 10], where R is a principal ideal domain. In this section, we will give an important auxiliary result of the main theorem in Proposition 3.1 and illustrate it with examples.

Lemma 3.1. Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs. Then each element in $\mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}'))$ $(n \geq 0)$ can be written in the form

$$\sum_{i=1}^{m} x_i \otimes y_i, \quad \deg(x_i) = p_i, \ \deg(y_i) = q_i, \ p_i + q_i = n$$

where x_i, y_i are linear combinations of hyperedges of $\mathcal{H}$ and $\mathcal{H}'$ respectively such that for each $1 \leq i \leq m$,

$$(x_i \otimes y_i) \in \mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}')).$$

proof. By (1), we know that

$$\mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}')) = (R(\mathcal{H}) \otimes R(\mathcal{H}'))_n \cap (\partial \otimes \partial')_n^{-1} (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}.$$

Let

$$g = r_1(\sigma_1 \otimes \tau_1) + \cdots + r_l(\sigma_l \otimes \tau_l)$$

be an element in

$$(R(\mathcal{H}) \otimes R(\mathcal{H}'))_n \cap (\partial \otimes \partial')_n^{-1} (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}$$

where $\sigma_i \in \mathcal{H}$, $\tau_i \in \mathcal{H}'$ and $r_i \in R$.

Notice that

$$(\partial \otimes \partial')_n(g) \in (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}$$

and

$$(\partial \otimes \partial')_n(\sigma_1 \otimes \tau_1) = \partial\sigma_1 \otimes \tau_1 + (-1)^{\deg(\sigma_1)}\sigma_1 \otimes \partial'\tau_1.$$

Consider the following two cases.

CASE 1. $\partial\sigma_1 \in R(\mathcal{H})_{p_1-1}$ and $\partial'\tau_1 \in R(\mathcal{H}')_{q_1-1}$. Then

$$\sigma_1 \otimes \tau_1 \in (R(\mathcal{H}) \otimes R(\mathcal{H}'))_n \cap (\partial \otimes \partial')_n^{-1}(R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}.$$

Set $x_1 = r_1\sigma_1$ and $y_1 = \tau_1$.

CASE 2. $\partial\sigma_1 \notin R(\mathcal{H})_{p_1-1}$ or $\partial'\tau_1 \notin R(\mathcal{H}')_{q_1-1}$. Without loss of generality, we can assume that $\partial\sigma_1 \notin R(\mathcal{H})_{p_1-1}$. Then there exists $0 \le k_1 \le p_1$ such that $d_{k_1}(\sigma_1) \notin R(\mathcal{H})_{p_1-1}$. Since $(\partial \otimes \partial')_n(g) \in (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}$, it follows that

$$\sum_{\{(j,k_j) | d_{k_j}(\sigma_j) = d_{k_1}(\sigma_1)\}} (-1)^{k_j} r_j = 0,$$

with $\tau_1 = \tau_j$, $p_1 = p_j$ and $q_1 = q_j$. Hence we get a linear combination $x_1 = (r_1\sigma_1 + r_j\sigma_j + \cdots)$ of finite hyperedges in $\mathcal{H}$ and $y_1 = \tau_1$.

By repeating the above process finite times and combing Case 1 and Case 2, the lemma follows.

We give an example to illustrate the form of element in $\mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}'))$.

Example 3.1. Let

$$\begin{aligned} \mathcal{H} &= \{\{v_1\}, \{v_3\}, \{v_1, v_2\}, \{v_2, v_3\}, \{v_1, v_3\}\}, \\ \mathcal{H}' &= \{\{w_2\}, \{w_3\}, \{w_1, w_2\}, \{w_2, w_3\}, \{w_1, w_3\}\} \end{aligned}$$

be two hypergraphs and $R = \mathbb{Z}$.

Let

$$g = \{v_1, v_2\} \otimes \{w_2, w_3\} + \{v_1, v_3\} \otimes \{w_1, w_3\} \quad + \{v_2, v_3\} \otimes \{w_2, w_3\} - \{v_1, v_3\} \otimes \{w_1, w_2\} + \{v_1, v_3\} \otimes \{w_2, w_3\}.$$

It can be directly verified that

$$\begin{aligned} g &\in \mathrm{Inf}_2(R(\mathcal{H}) \otimes R(\mathcal{H}')), \\ \{v_1, v_3\} \otimes \{w_2, w_3\} &\in \mathrm{Inf}_2(R(\mathcal{H}) \otimes R(\mathcal{H}')) \end{aligned}$$

while none of

$$\begin{aligned} &\{v_1, v_2\} \otimes \{w_2, w_3\}, \quad \{v_1, v_3\} \otimes \{w_1, w_3\}, \\ &\{v_2, v_3\} \otimes \{w_2, w_3\}, \quad \{v_1, v_3\} \otimes \{w_1, w_2\} \end{aligned}$$

is in $\mathrm{Inf}_2(R(\mathcal{H}) \otimes R(\mathcal{H}'))$. But we can express g as

$$g = (\{v_1, v_2\} + \{v_2, v_3\}) \otimes \{w_2, w_3\} + \{v_1, v_3\} \otimes (\{w_1, w_3\} - \{w_1, w_2\}) + \{v_1, v_3\} \otimes \{w_2, w_3\},$$

in which each term

$$\begin{aligned} &(\{v_1, v_2\} + \{v_2, v_3\}]) \otimes \{w_2, w_3\}, \\ \{v_1, v_3\} \otimes (\{w_1, w_3\} - \{w_1, w_2\}), \;\; &\{v_1, v_3\} \otimes \{w_2, w_3\} \end{aligned}$$

is in $\mathrm{Inf}_2(R(\mathcal{H}) \otimes R(\mathcal{H}'))$.

Proposition 3.1. Let $\mathcal{H}$, $\mathcal{H}'$ be two hypergraphs. Then for any $n \ge 0$, we have

$$\mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}')) = (\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}'))_n.$$

Proof. By substituting D and D' with $\mathrm{Inf}_*(R(\mathcal{H}))$ and $\mathrm{Inf}_*(R(\mathcal{H}'))$ in Lemma 2.1 respectively, we have that

$$(\partial \otimes \partial')_n((\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}')))_n = \sum \Big(\partial_p\big(\mathrm{Inf}_p(R(\mathcal{H}))\big) \otimes \mathrm{Inf}_q(R(\mathcal{H}')) + \mathrm{Inf}_p(R(\mathcal{H})) \otimes \partial'_q\big(\mathrm{Inf}_q(R(\mathcal{H}'))\big)\Big) \subseteq (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}.$$

Moreover,

$$(\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}')))_n \subseteq (R(\mathcal{H}) \otimes R(\mathcal{H}'))_n.$$

Hence

$$(\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}')))_n \subseteq (R(\mathcal{H}) \otimes R(\mathcal{H}'))_n \cap (\partial \otimes \partial')_n(R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1},$$

which implies that

$$(\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}')))_n \subseteq \mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}')).$$

On the other hand, for each term

$$x \otimes y \in \mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}')),\ \deg(x) = p,\ \deg(y) = q,\ p + q = n,$$

we have that

$$(\partial \otimes \partial')_n(x \otimes y) = ((\partial_p x) \otimes y + (-1)^p x \otimes (\partial'_q y)) \in (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}.$$

Then

$$(\partial_p x) \otimes y \in (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1},\ x \otimes (\partial'_q y) \in (R(\mathcal{H}) \otimes R(\mathcal{H}'))_{n-1}.$$

Hence

$$x \in R(\mathcal{H})_p \cap \partial_p^{-1} R(\mathcal{H})_{p-1}, \quad y \in R(\mathcal{H}')_q \cap \partial'^{-1}_q R(\mathcal{H}')_{q-1}$$

and

$$(x \otimes y) \in (\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}')))_n.$$

Hence, by Lemma 3.1,

$$\mathrm{Inf}_n(R(\mathcal{H}) \otimes R(\mathcal{H}')) \subseteq (\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}')))_n.$$

The proposition is proved.

We give an example to illustrate Proposition 3.1.

Example 3.2. Consider the hypergraphs $\mathcal{H}$ and $\mathcal{H}'$ given in Example 3.1. Then

$$\begin{aligned}
\mathrm{Inf}_*(R(\mathcal{H})) &= R\big\{\{v_1\}, \{v_3\}, \{v_1, v_3\}, \{\{v_1, v_2\} - \{v_2, v_3\}\}\big\} \\
\mathrm{Inf}_*(R(\mathcal{H}')) &= R\big\{\{w_2\}, \{w_3\}, \{w_2, w_3\}, \{\{w_1 w_2\} - \{w_1, w_3\}\}\big\} \\
\mathrm{Inf}_*(R(\mathcal{H})) \otimes \mathrm{Inf}_*(R(\mathcal{H}')) &= R\Big\{\{v_1\} \otimes \{w_2\}, \{v_1\} \otimes \{w_3\}, \{v_3\} \otimes \{w_2\}, \\
&\quad \{v_3\} \otimes \{w_3\}, \{v_1, v_3\} \otimes \{w_2, w_3\}, \{v_1\} \otimes \{w_2, w_3\}, \\
&\quad \{v_1, v_3\} \otimes \big(\{w_1, w_2\} - \{w_1, w_3\}\big), \\
&\quad \big(\{v_1, v_2\} + \{v_2, v_3\}\big) \otimes \big(\{w_1, w_2\} - \{w_1, w_3\}\big), \\
&\quad \{v_1\} \otimes \big(\{w_1, w_2\} - \{w_1, w_3\}\big), \{v_3\} \otimes \{w_2, w_3\}, \\
&\quad \{v_3\} \otimes \big(\{w_1, w_2\} - \{w_1, w_3\}\big), \{v_1, v_3\} \otimes \{w_2\}, \\
&\quad \{v_1, v_3\} \otimes \{w_3\}, \big(\{v_1, v_2\} + \{v_2, v_3\}\big) \otimes \{w_2\}, \big(\{v_1, v_2\} + \{v_2, v_3\}\big) \otimes \{w_3\}\Big\}
\end{aligned}$$

$$\begin{aligned}
R(\mathcal{H})\otimes R(\mathcal{H}') \;=\; & R\Big\{\{v_1\}\otimes\{w_2\},\{v_1\}\otimes\{w_3\},\{v_3\}\otimes\{w_2\},\\
& \{v_3\}\otimes\{w_3\},\{v_1,v_2\}\otimes\{w_2\},\{v_1,v_3\}\otimes\{w_3\},\\
& \{v_1\}\otimes\{w_1,w_2\},\{v_1\}\otimes\{w_2,w_3\},\{v_1\}\otimes\{w_1,w_3\},\\
& \{v_3\}\otimes\{w_1,w_2\},\{v_3\}\otimes\{w_2,w_3\},\{v_3\}\otimes\{w_1,w_3\},\\
& \{v_1,v_2\}\otimes\{w_1,w_2\},\{v_1,v_2\}\otimes\{w_2,w_3\},\\
& \{v_1,v_2\}\otimes\{w_1,w_3\},\{v_2,v_3\}\otimes\{w_1,w_2\},\\
& \{v_2,v_3\}\otimes\{w_2,w_3\},\{v_2,v_3\}\otimes\{w_1,w_3\},\\
& \{v_1,v_3\}\otimes\{w_1,w_2\},\{v_1,v_3\}\otimes\{w_2,w_3\},\{v_1,v_3\}\otimes\{w_1,w_3\}\Big\}
\end{aligned}$$

$$\begin{aligned}
\mathrm{Inf}_*(R(\mathcal{H})\otimes R(\mathcal{H}')) \;=\; & R\Big\{\{v_1\}\otimes\{w_2\},\{v_1\}\otimes\{w_3\},\{v_3\}\otimes\{w_2\},\\
& \{v_3\}\otimes\{w_3\},\big(\{v_1,v_2\}+\{v_2,v_3\}\big)\otimes\{w_2\},\\
& \big(\{v_1,v_2\}+\{v_2,v_3\}\big)\otimes\{w_3\},\{v_1,v_3\}\otimes\{w_2\},\\
& \{v_1,v_3\}\otimes\{w_3\},\{v_3\}\otimes\{w_2,w_3\},\{v_1\}\otimes\{w_2,w_3\}\\
& \{v_1\}\otimes\big(\{w_1,w_2\}-\{w_1,w_3\}\big),\{v_3\}\otimes\big(\{w_1,w_2\}-\{w_1,w_3\}\big)\\
& \big(\{v_1,v_2\}+\{v_2,v_3\}\big)\otimes\big(\{w_1,w_2\}-\{w_1,w_3\}\big)\Big\}
\end{aligned}$$

Hence,

$$\mathrm{Inf}_*(R(\mathcal{H}))\otimes\mathrm{Inf}_*(R(\mathcal{H}'))=\mathrm{Inf}_*(R(\mathcal{H})\otimes R(\mathcal{H}')).$$

4. A Künneth Formula for the Embedded Homology

By using the embedded homology, we prove a Künneth formula for the tensor product of graded abelian subgroups of chain complexes, corresponding to Theorem 4.2.

Theorem 4.1 (Algebraic Künneth Formula). ([7, Theorem 3B.5]) Let R be a principal ideal domain, and let C_*, C'_* be chain complexes of free R-module. Then there is a natural exact sequence

$$0\to\bigoplus_{p+q=n}H_p(C_*)\otimes_R H_q(C'_*)\to H_n(C_*\otimes C'_*)\to\bigoplus_{p+q=n}\mathrm{Tor}_R(H_p(C_*),H_{q-1}(C'_*))\to 0.$$

Theorem 4.2. Let R be a principal ideal domain. Let $\mathcal{H}$ and $\mathcal{H}'$ be two hypergraphs. Then there is a short exact sequence

$$0\longrightarrow(H_*(\mathcal{H})\otimes_R H_*(\mathcal{H}'))_n\longrightarrow H_n(\mathrm{Inf}_*(R(\mathcal{H})\otimes R(\mathcal{H}')))\longrightarrow\bigoplus_i\mathrm{Tor}_R(H_i(\mathcal{H}),H_{n-i-1}(\mathcal{H}'))\longrightarrow 0.$$

And this sequence splits.

Proof. By Theorem 4.1, we have a short exact sequence

$$\begin{aligned}
0\longrightarrow&(H_*(\mathrm{Inf}_*(D,C))\otimes_R H_*(\mathrm{Inf}_*(D',C')))_n\xrightarrow{\varphi}H_n(\mathrm{Inf}_*(D,C)\otimes_R\mathrm{Inf}_*(D',C'))\longrightarrow\\
&\bigoplus_i\mathrm{Tor}_R(H_i(\mathrm{Inf}_*(D,C)),H_{n-i-1}(\mathrm{Inf}_*(D',C')))\longrightarrow 0.
\end{aligned}\tag{5}$$

And this sequence splits.

By Proposition 3.1,

$$H_n(\mathrm{Inf}_*(R(\mathcal{H}))\otimes\mathrm{Inf}_*(R(\mathcal{H}')))=H_n(\mathrm{Inf}_*(R(\mathcal{H})\otimes R(\mathcal{H}'))).\tag{6}$$

The theorem follows from (5) and (6).

Let R be a field $\mathbb{F}$. The next corollary follows from Theorem 4.2.

Corollary 4.1. Suppose the chain complexes C and C' consist of graded vector spaces over a field $\mathbb{F}$. Let D and D' be graded vector subspaces of C and C' respectively. Then

$$H_*(\mathrm{Inf}_*(R(\mathcal{H})\otimes R(\mathcal{H}')))\cong H_*(\mathcal{H})\otimes_R H_*(\mathcal{H}').$$

5. Conclusion

From above discussion, we know that the critical proof of Theorem 4.2 is that Proposition 3.1 holds, which depends on Lemma 3.1. Let R be a principal ideal domain. Let C and C' be chain complexes consisting of graded free R-modules. Let D and D' be graded sub-R-modules of C and C' respectively and the generator sets of D and D' be subsets of the generator sets of C and C' respectively. Then

$$\mathrm{Inf}_*(D \otimes D', C \otimes C') = \mathrm{Inf}_*(D, C) \otimes \mathrm{Inf}_*(D', C'),$$

which will be proved in our another paper. Hence, the analogue of Künneth formula of hypergraphs given in this paper can be extended to the Künneth formula of embedded homology of graded abelian groups of chain complexes.

Moreover, it is proved that the path homology of digraphs is consistent with the embedded homology of digraphs [15]. A. Grigor'yan, Y. Lin, Y. Muranov and S. T. Yau studied the the Künneth formula for the path homology (with coefficients in a field) of digraphs [5, 6]. Therefore, we can get the Künneth formula for digraphs with coefficients in a principal ideal domain.

Acknowledgements

The author would like to express her deep gratitude to the reviewer(s) for their careful reading, valuable comments, and helpful suggestions.

The author is supported by Science and Technology Project of Hebei Education Department (QN2019333), the Natural Fund of Cangzhou Science and Technology Bureau (No.197000002) and a Project of Cangzhou Normal University (No.xnjjl1902).

References

[1] C. Berge, *Graphs and hypergraphs*. North-Holland Mathematical Library, Amsterdam, 1973.

[2] S. Bressan, J. Li, S. Ren and J. Wu, *The embedded homology of hypergraphs and applications*. Asian J. Math. 23 (3) (2019), 479-500.

[3] F. R. K. Chung and R. L. Graham, *Cohomological aspects of hypergraphs*. Trans. Amer. Math. Soc. 334 (1) (1992), 365-388.

[4] E. Emtander, *Betti numbers of hypergraphs*. Commun. Algebra 37 (5), (2009), 1545-1571.

[5] A. Grigor'yan, Y. Lin, Y. Muranov and S.T. Yau, *Homologies of path complexes and digraphs*. arXiv (2012). http://arxiv.org/abs/1207.2834.

[6] A. Grigor'yan, Y. Muranov and S. T. Yau, *Homologies of digraphs and Künneth formulas*. Commun. Anal. Geom. 25 (5) (2017), 969-1018.

[7] A. Hatcher, *Algebraic topology*. Cambridge University Press, Cambridge, 2001.

[8] J. Johnson, *Hyper-networks of complex systems*, in *Complex Sciences*, Series Lecture Notes of the Institute for Computer Sciences, Social Informatics and Telecommunications Engineering 4, (2009), 364-375.

[9] J. P. May, *Simplicial objects in algebraic topology*. Van Nostrand, 1967 (reprinted by University of Chicago Press).

[10] J. W. Milnor, *The geometric realization of a semi-simplicial complex*. Ann. of Math. 65 (2) (1957), 357-362.

[11] S. Klamt, U. Haus and F. Theis, *Hypergraphs and cellular networks*. PLoS Computational Biology 5 (5) (2009), Article number e1000385.

[12] R. Lung, N. Gaskó, M.A. Suciu, *A hypergraph model for representing scientific output*. Scientometrics 117 (3) (2018), 1361-1379.

[13] A. D. Parks and S. L. Lipscomb, *Homology and hypergraph acyclicity: a combinatorial invariant for hypergraphs*. Naval Surface Warfare Center, 1991.

[14] Z. Meng and K. Xia, *Persistent spectral based machine learning (PerSpect ML) for drug design*. arXiv https://arxiv.org/abs/2002.00582, preprint.

[15] Chong Wang, Shiquan Ren, *Simplicial Descriptions for Digraphs and Their Path Homology From the Point of Δ-sets*(in Chiniese). Mathematics in Practice and Theory, 49 (22) (2019), 238-247.